Orbiting Strings in AdS_5 Black Hole Backgrounds

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Abstract

Following Armoni, Barbon and Petkou [hep-th/0205280], we study the orbits of classical closed strings in AdS_5 black hole backgrounds. A geodesic equation describing the orbits of classical closed strings in arbitrary spacetimes is derived. We consider the string configuration given by the ansatz in [hep-th/0205280] for the AdS_5 Schwarzschild, Kerr-AdS_5 and AdS_5-Reissner-Nordström black hole backgrounds. We find that such a string configuration results in the classical closed string to orbit around the black hole horizons for the AdS_5 Schwarzschild, Kerr-AdS_5 and AdS_5-Reissner-Nordström black hole backgrounds.

Contents

1 Introduction

General Relativity describes gravity as the curvature of spacetime [1, 2, 3]. Mathematically we may describe the behaviour of spacetime using Einstein's equations with cosmological constant, which are given by [2, 3]

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, Λ is Einstein's cosmological constant, $T_{\alpha\beta}$ is the energy-stress tensor, $g_{\alpha\beta}$ is the metric of the spacetime and in units we have chosen $c = 8\pi G = 1$ as seen in [2]. For vacuum solutions Einstein's equations simply reduce to $G_{\alpha\beta} = 0$ [3], without the cosmological constant. Solving for exact solutions from Einstein's equations is an extremely difficult task; however, this does not prevent us from showing that certain suspected spacetimes do in fact satisfy Einstein's equations (1).

An example of a spacetime which satisfies the Einstein equations with a cosmological term is the AdS_5 geometry, which can be described in global coordinates by the metric taken from [4]

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \tag{2}$$

where R is the AdS radius and $d\Omega_3^2$ is the metric on the round unit 3-sphere given by

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2$$

AdS or Anti-de Sitter space can be visualized as a hyperboloid with two spatial dimensions suppressed [5] and can be described as a maximally symmetric spacetime with a negative cosmological constant [1]. Spatially AdS is infinite; however, one is able to define a boundary at infinity which we shall simply refer to as the AdS boundary [1]. AdS is not a physically sensible spacetime as it contains closed timelike curves [5]; one would then be well justified in asking the question, "Why then should we concern ourselves with such spacetime geometries?".

Anti-de Sitter spacetimes are of particular interest from other spacetimes due to the implications associated with the AdS/CFT correspondence, otherwise known as the Maldacena conjecture. The Maldacena conjecture proposes that a string theory defined on a spacetime $\mathcal{M} = \mathrm{AdS}_n \times S^m$ with dimension $q = n + m$, is to be equivalent to a certain supersymmetric Yang-Mills theory defined on a spacetime $\mathcal{N}$ with dimension p, such that $p = n - 1$ [6]. Where $\mathcal{M}$ is the product of an AdS geometry with dimension n and a sphere S with dimension m.

Thanks to the Bekenstein-Hawking entropy formula we know that black holes have a well defined entropy $\mathcal{S}$ [2, 3]. From quantum theory we know that for an entropy $\mathcal{S}$ there is a corresponding large number of quantum states that describe the microscopic behavior of the black holes [1]. Therefore the Maldacena conjecture is certainly a promising weapon for the attack on one of the greatest problems in physics, a quantum understanding of gravity. A suitable stadium to conduct such an attack would be to consider AdS black holes as there are many questions surrounding the quantum nature of black holes.

From further research into the AdS/CFT correspondence a dictionary has been developed whereby geometrical notions present within AdS can be mapped to dual observables within the CFT. According to [7], string solutions within AdS correspond to heavy operators which are made up of a large number of fields. Thus according to [4], orbiting strings around the outside of black holes should describe physical states. Therefore understanding the orbits of strings around AdS_5 black holes could potentially reveal interesting relations between the orbits and their duals at finite temperature in $\mathcal{N} = 4$ SYM. Thus, for the purposes of this study we will consider the orbits of classical closed strings around the outside of various types of AdS_5 black holes.

This manuscript has been arranged as follows: Section 2 we study the derivation of the geodesic equation for a point particle in a curved background. We then use a similar argument to derive a geodesic equation for the case of a closed string. Section 3 we then study the orbits of a classical closed string around a AdS_5 Schwarzschild black hole background as in [4]. Subsequently, in section 4 we show that the constraint equations obtained in section 3 are in fact logically equivalent to the geodesic equation obtained in Section 2. Section 5 we show how to compute the energy and angular momentum of strings in curved backgrounds. Section 6 we consider an alternative string configuration namely the pulsating string around the AdS_5 Schwarzschild black hole background. Finally, sections 7 and 8 we consider the string configuration given in [4], but in the Kerr-AdS_5 and AdS_5-Reissner-Nördstrom black hole backgrounds.

2 Geodesic Equations

2.1 Point Particle

Consider a relativistic point particle that is moving through a D-dimensional curved spacetime with a metric $g_{\mu\nu}$. As the particle moves through this spacetime it traces what is called a world-line [8, 9]. We shall denote this world-line by $X^\mu(\tau)$, where we have parametrized the world-line by τ. The world-line traced by the particle can be described by a geodesic equation [1, 8]. Thus, using the variational method we make use of the relativistic action given by

$$S_R = -m \int ds \tag{3}$$

where we are integrating over the infinitesimal line element ds. This line element can be described by

$$ds = \sqrt{-g_{\mu\nu} dX^\mu dX^\nu} \tag{4}$$

Now, since we parametrize the world line by the parameter τ, we integrate over this parameter obtaining the result

$$S_R = -m \int \sqrt{-g_{\mu\nu} dX^\mu dX^\nu} d\tau \tag{5}$$

We wish to vary this action with respect to the world-line $X^\mu(\tau)$ as we wish to search for the minimal path of $X^\mu(\tau)$. However; due to the presence of the square root, varying this action proves to be a challenging task. Therefore we consider an alternative action that is equivalent to the relativistic action which we denote by S_E. This action is given by [1, 8]

$$S_E = \frac{1}{2} \int d\tau (e^{-1} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - m^2 e) \tag{6}$$

where $e(\tau)$ is the auxiliary field. This action is only equivalent to the relativistic action at the classical level. The proof that this action is equivalent to the relativistic action is provided Appendix A.

Since this action is invariant under reparametrizations, this allows us to gauge fix the auxiliary field to $e(\tau) = 1$ [1]. Now we take the variation of this result with respect to $X^\mu(\tau)$

$$\delta S_E = -\frac{1}{2} \int d\tau (\partial_\rho g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \delta X^\rho + 2 g_{\mu\nu} \delta \dot{X}^\mu \dot{X}^\nu) \tag{7}$$

Now we integrate the second term by parts and we find that

$$\int d\tau (g_{\mu\nu} \delta \dot{X}^\mu \dot{X}^\nu) = \delta X^\mu g_{\mu\nu} \dot{X}^\mu \Big|_{-\infty}^{+\infty} - \int d\tau (\partial_\kappa g_{\mu\nu} \dot{X}^\kappa \dot{X}^\nu + g_{\mu\nu} \ddot{X}^\nu) \delta X^\mu \tag{8}$$

The first term of this result reduces to zero due to the fact that $\delta X \to 0$ as $\tau \to \pm\infty$. Substituting equation (8) into equation (7)

$$\delta S_E = \int d\tau (-\partial_\mu g_{\rho\nu} \dot{X}^\rho \dot{X}^\nu + 2\partial_\kappa g_{\mu\nu} \dot{X}^\kappa \dot{X}^\nu + 2 g_{\mu\nu} \ddot{X}^\nu) \delta X^\mu \tag{9}$$

Now we set the variation $\frac{\delta S_E}{\delta X^\mu} = 0$. This allows us to obtain the minimal equation [10]. Thus the geodesic equation for a relativistic point particle is

$$-\partial_\mu g_{\rho\nu}\dot{X}^\rho\dot{X}^\nu + 2\partial_\kappa g_{\mu\nu}\dot{X}^\kappa\dot{X}^\nu + 2g_{\mu\nu}\ddot{X}^\nu = 0 \tag{10}$$

relabeling indices and multiplying through by the inverse metric

$$\ddot{X}^\mu + g^{\lambda\mu}(\partial_\rho g_{\lambda\kappa}\dot{X}^\rho\dot{X}^\kappa - \frac{1}{2}\partial_\lambda g_{\rho\kappa}\dot{X}^\rho\dot{X}^\kappa) = 0 \tag{11}$$

This expression invites the simplification by using the Christoffel symbols. We finally obtain the well known geodesic equation for a relativistic point particle

$$\ddot{X}^\mu + \Gamma^\mu_{\rho\kappa}\dot{X}^\rho\dot{X}^\kappa = 0 \tag{12}$$

2.2 Closed String

Consider a closed string moving through a D-dimensional curved spacetime with metric $g_{\mu\nu}$. As a string moves through spacetime its trajectory traces a world sheet, which is denoted by $X^\mu(\sigma,\tau)$ and is parameterized by the coordinates τ and σ. This can be seen in figure 1 below.

A closed string is a string that has been folded such that its endpoints are connected like a hula-hoop. Due to the physical structure of the string we impose the periodic boundary conditions on the string, an illustration of these boundary conditions can be seen in figure 1 (b).

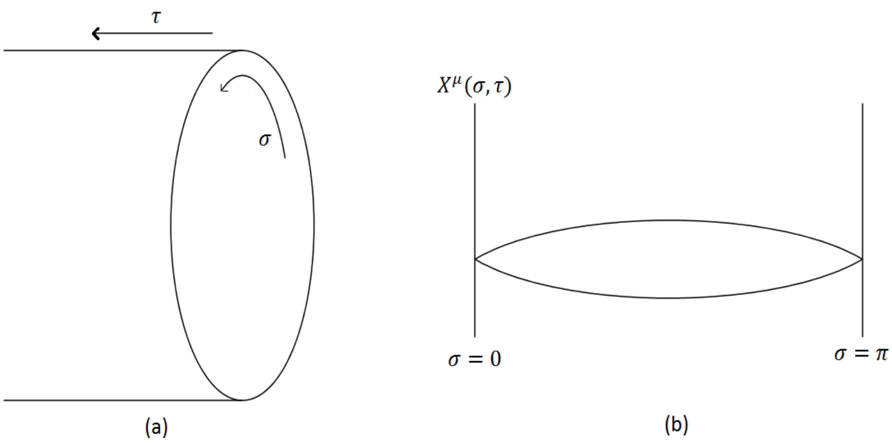


Figure 1: (a) Illustration of the choice of parametrization on the world sheet and (b) shows the periodic boundary conditions on the closed string.

In this variational problem we make use of the Nambu-Goto action, which is given by [1, 4, 11]

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)} \tag{13}$$

where α' is the tension of the string, $g_{\mu\nu}$ is the embedded space-time metric and $\alpha, \beta = \sigma, \tau$.

Due to the presence of the square root in the Nambu-Goto action, varying the action proves to be a difficult task. Thus we introduce the Polyakov action which is proven to be equivalent to the Nambu-Goto action - provided in Appendix A. The Polyakov action is given by [11]

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \tag{14}$$

where $h_{\alpha\beta}$ is the auxiliary world sheet metric, with $h_{\alpha\beta} = h_{\alpha\beta}(\sigma, \tau)$ and $h = det(h_{\alpha\beta})$. It must be noted that this equivalence is only true at the classical level [1].

As seen in Figure 1(b) above we have chosen the periodic boundary conditions for the closed string, therefore it is required that the σ coordinate be bounded by $0 \leq \sigma \leq 2\pi$. Starting with the Polyakov action (14) we use the fact that reparametrization invariance of the world sheet $X^\mu(\sigma, \tau)$ allows the auxiliary world sheet metric $h_{\alpha\beta}$ to be gauge fixed. We gauge fix the auxiliary field to

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

This is called the conformal gauge [11]. It must be noted that this choice of gauge fixing restricts the world sheet to one with vanishing Euler characteristic [1].
Now the Polyakov action becomes

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\eta^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \tag{16}$$

which can then be rewritten as

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau g_{\mu\nu}(-\dot{X}^\mu\dot{X}^\nu + X'^\mu X'^\nu) \tag{17}$$

where we denote $\dot{X} = \frac{\partial X}{\partial\tau}$ and $X' = \frac{\partial X}{\partial\sigma}$. We vary the Polyakov action with respect to $X^\mu(\sigma, \tau)$ and subsequently set $\frac{\delta S_\sigma}{\delta X^\mu} = 0$ to obtain the geodesic equation. Therefore the variation of equation (17) with respect to $X^\mu(\sigma,\tau)$ gives

$$\delta S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau(-\partial_\kappa g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu\delta X^\kappa - 2g_{\mu\nu}\delta\dot{X}^\mu\dot{X}^\nu \\ +\partial_\kappa g_{\mu\nu}X'^\mu X'^\nu\delta X^\kappa + 2g_{\mu\nu}\delta X'^\mu X'^\nu) \tag{18}$$

To obtain this equation in terms of δX^μ we integrate the second and fourth expression by parts. We integrate the second expression by parts in terms of τ and we find that

$$\int d\tau g_{\mu\nu}\delta\dot{X}^\mu\dot{X}^\nu = \delta X^\mu g_{\mu\nu}\dot{X}^\nu\Big|_{-\infty}^{+\infty} - \int d\tau(\partial_\kappa g_{\mu\nu}\dot{X}^\kappa\dot{X}^\nu + g_{\mu\nu}\ddot{X}^\nu)\delta X^\mu \tag{19}$$

the first term of this result reduces to zero. This is because $\delta X \to 0$ as $\tau \to \pm\infty$. On the other hand we integrate the fourth expression by parts in terms of σ

$$\int d\sigma g_{\mu\nu}\delta X'^\mu X'^\nu = \delta X^\mu g_{\mu\nu}X'^\nu\Big|_0^{2\pi} - \int d\sigma(\partial_\kappa g_{\mu\nu}X'^\mu X'^\nu + g_{\mu\nu}X''^\nu)\delta X^\mu \tag{20}$$

Since we are considering a closed string it is true that $X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau)$, therefore the first term of equation (20) reduces to zero as we are integrating from 0 to 2π.
Substituting the results of equation (19) and (20) into equation (18) we have

$$\frac{\delta S_\sigma}{\delta X^\mu} = \int d\sigma d\tau(-\partial_\mu g_{\rho\nu}\dot{X}^\rho\dot{X}^\nu + 2\partial_\kappa g_{\mu\nu}\dot{X}^\kappa\dot{X}^\nu + 2g_{\mu\nu}\ddot{X}^\nu \\ +\partial_\mu g_{\rho\nu}X'^\rho X'^\nu - 2\partial_\kappa g_{\mu\nu}X'^\kappa X'^\nu - 2g_{\mu\nu}X''^\nu) \tag{21}$$

Now that we have this result, we set the variation $\frac{\delta S_\sigma}{\delta X^\mu} = 0$ in order to obtain the minimal equation. Therefore we are simply left with

$$-\partial_\mu g_{\rho\nu}\dot{X}^\rho\dot{X}^\nu + 2\partial_\kappa g_{\mu\nu}\dot{X}^\kappa\dot{X}^\nu + 2g_{\mu\nu}\ddot{X}^\nu \\ +\partial_\mu g_{\rho\nu}X'^\rho X'^\nu - 2\partial_\kappa g_{\mu\nu}X'^\kappa X'^\nu - 2g_{\mu\nu}X''^\nu = 0 \tag{22}$$

this is the geodesic equation for a classical string in a curved background. However, we may simplify this equation to the more compact form

$$\ddot{X}^\mu - X''^\mu + \Gamma^\mu_{\rho\lambda}(\dot{X}^\rho\dot{X}^\lambda - X'^\rho X'^\lambda) = 0 \tag{23}$$

where we have simply introduced the Christoffel symbol to obtain this result. These are the geodesic equations that describe the motion of closed classical strings in arbitrary backgrounds. From this equation, we can see that the motion of the string is completely determined by the geometry of the spacetime in which it is present in.

3 AdS$_5$ Schwarzschild black hole background

We are now concerned with the behavior of classical closed strings in the AdS$_5$ Schwarzschild black hole background, given by the metric [4] as

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) \tag{24}$$

where the AdS$_5$ black hole, $f(r)$ is given by

$$f(r) = 1 + \frac{r^2}{R^2} - \frac{M}{r^2}$$

where M is related to the mass of the AdS$_5$ black hole. We wish to consider a similar string configuration as seen in [4], subsequently we wish to verify some of the rough features of the orbits for the AdS_5 black hole case.

The physical orientation of the string in the respective spacetime is encoded within the ansatz, the ansatz takes the general form $X^\mu(\tau,\sigma) = (t, r, \theta, \phi_1.\phi_2)$, where each variable may depend on τ or σ. In [4] they consider a string that is extended in the radial r direction with a constant angular velocity $\dot{\phi}_1(\tau,\sigma) = \omega$. The ansatz for such a string configuration is given by

$$X^\mu(\tau,\sigma) = (\tau, r(\sigma), \frac{\pi}{2}, \omega\tau, 0) \tag{25}$$

where the ansatz is of the form $X^\mu(\tau,\sigma) = (t(\tau,\sigma), r(\tau,\sigma), \theta(\tau,\sigma), \phi_1(\tau,\sigma), \phi_2(\tau,\sigma))$.

We can see from the choice that $r(\tau,\sigma) = r(\sigma)$ that the "face" of the string does not point in the direction of the center of the black hole $r = 0$. Therefore we expect to find a point r_{min} on the string that is closest to the center of the black hole and a point r_{max} on the string that is farthest from the black hole compared to any other points on the string.

An idea of the type of orientation can be seen in Figure 3. Now using the Virasoro constraint equation (86); and the AdS_5 black hole metric (24), we find that the constraint equations are

$$g_{tt}(\dot{X}^t)^2 + g_{rr}(X'^r)^2 + g_{\phi_1\phi_1}(\dot{X}^{\phi_1})^2 = 0 \tag{26}$$

which gives

$$\left(\frac{dr}{d\sigma}\right)^2 = \frac{(r^4(1-\omega^2) + r^2 - M)(r^4 + r^2 - M)}{r^4} \tag{27}$$

where as in [4], we have taken $R = 1$.

Given this result, we need to understand the behavior of r as a function of σ. As σ varies we expect to traverse the string itself, as we parametrized the closed string

by σ as seen in Figure 1 (a). Therefore we expect to see $r(\sigma)$ with a sort of oscillatory behavior with stationary points at r_{min} and r_{max}.
This idea can be described by the graph below

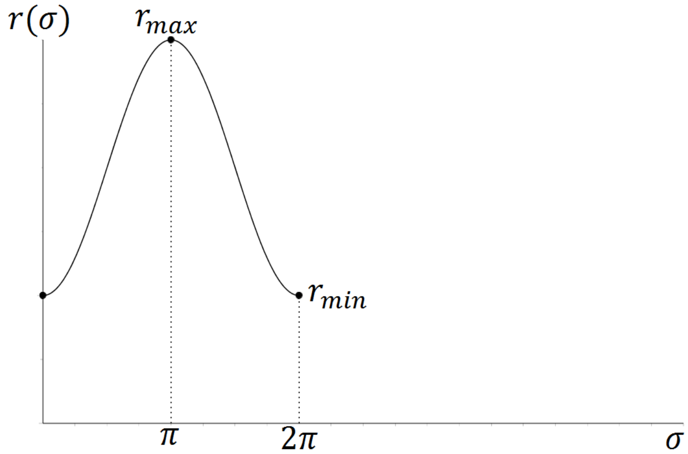


Figure 2: General behavior of $r(\sigma)$ with stationary points r_{min} and r_{max}.

In order to find the stationary points r_{min} and r_{max}, we set $\frac{dr}{d\sigma} = 0$ in equation (27). Then we find that

$$r^4(1-\omega^2) + r^2 - M = 0 \tag{28}$$

To solve this equation we make use of the quadratic formula and set $a = r^2$. Thus

$$a = \frac{1 \pm \sqrt{1 + 4M(1-\omega^2)}}{2(1-\omega^2)} \tag{29}$$

Thus, solving for r_{min} and r_{max}

$$r^2_{max} = \frac{1 + \sqrt{1 - 4M(\omega^2 - 1)}}{2(\omega^2 - 1)} \qquad r^2_{min} = \frac{1 - \sqrt{1 - 4M(\omega^2 - 1)}}{2(\omega^2 - 1)} \tag{30}$$

with $r_{min} < r_{max}$. Due to the fact that these two equations must be real and positive [4] we can thus impose restrictions on the possible values for ω. So the square root terms must be positive and we find that

$$\omega^2 \leq \frac{1}{4M} + 1 \tag{31}$$

Furthermore when we set $\frac{dr}{d\sigma} = 0$ in equation (27), we also found that

$$r^4 + r^2 - M = 0 \tag{32}$$

The roots of this equation give the horizon of the black hole [4], which we denote by r_H. Firstly we make the substitution $\alpha = r^2$ into equation (32) thus allowing us to use the quadratic formula. We obtain the result

$$\alpha = \frac{-1 \pm \sqrt{1+4M}}{2} \tag{33}$$

Therefore we are left with

$$r_H^2 = \frac{-1 + \sqrt{1+4M}}{2} \qquad -r^2 = \frac{-1 - \sqrt{1+4M}}{2} \tag{34}$$

where as expected we find that $r_H < r_{min}$.

The orientation of the closed string with respect to the horizon of the black hole r_H can be seen in Figure 3 below. It is important to emphasize that this closed string is in fact folded, due to the fact that r depends on σ and $r(\sigma) = r(\sigma + 2\pi)$.

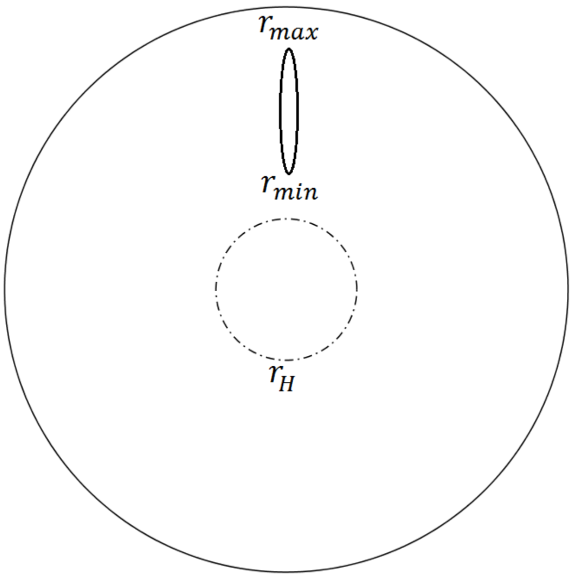


Figure 3: Schematic of a classical folded closed string in AdS_5 Schwarchild black hole background

Obviously figure 3 is not accurate, it simply serves as a tool to help us visualize the orientation of the folded closed string with respect to the horizon of the black hole r_H. We see that the closed string will orbit around the horizon and not around the center $r = 0$ of the black hole.

4 Consistency with Virasoro constraints

Since we have verified the Virasoro constraint equation that was obtained in [4], we now wish to show that the geodesic equation that we obtained for the case of a classical closed string (23) is in fact consistent with this constraint equation (27). To do this we consider the case where $\mu = r$ in equation (23), then we are required to show that

$$\ddot{X}^r - X''^r + \Gamma^r_{\rho\lambda}(\dot{X}^\rho \dot{X}^\lambda - X'^\rho X'^\lambda) \tag{35}$$

is equal to zero using constraint equation (27).

Now, using the ansatz (25) we find that expression (35) becomes

$$-X''^r + \Gamma^r_{tt}((\dot{X}^t)^2 - (X'^t)^2) + \Gamma^r_{rr}((\dot{X}^r)^2 - (X'^r)^2) + \Gamma^r_{\phi_1\phi_1}((\dot{X}^{\phi_1})^2 - (X'^{\phi_1})^2) \tag{36}$$

To obtain these Christoffel symbols we make use of Maple Soft 2015. The Maple code used to determine these Christoffel symbols can be found in Appendix B. Therefore, this equation can then be written as

$$-\frac{d^2r}{d\sigma^2} - \frac{(-r^2 - r^4 + M)(r^4 + M)}{r^5}$$

$$-\frac{(r^4 + M)}{r(-r^2 - r^4 + M)}\left(\frac{dr}{d\sigma}\right)^2 + \frac{(-r^2 - r^4 + M)}{r}\omega^2 = 0 \tag{37}$$

Now we differentiate the constraint equation (27) in terms of σ in order to determine $\frac{d^2r}{d\sigma^2}$. And we find that

$$\frac{d^2r}{d\sigma^2} = -\frac{2r^8\omega^2 - 2r^8 + r^6\omega^2 - 2r^6 - 2Mr^2 + 2M^2}{r^5} \tag{38}$$

Substituting the result of this equation with the constraint equation (27) into equation (37), we find that equation (37) vanishes.

Therefore the geodesic equation (23) is in fact consistent with the Virasoro constraint equation (27). The reader may then jump to ask the question, "Why then do we use the Virasoro constraint equations as opposed to the geodesic equations (27)?". This is an issue that we shall address in the conclusions.

5 Energy and angular momentum of a closed string

As seen in [4] the energy and angular momentum of a closed string is of great interest. In this section we shall provide an explanation as to how to compute the energy and angular momentum for the string configuration (25) in the AdS_5 Schwarzschild black hole background.

5.1 Energy

In order to calculate the energy of a closed string we start by considering the energy for a point on the string denoted ϵ. Then the total energy of a string is simply given by the integral

$$E = \int_{r_{min}}^{r_{max}} \epsilon d\sigma \tag{39}$$

where we are integrating over all points on the string in terms of $d\sigma$. Now in order to determine the energy for a point on the string we need to determine the conjugate momentum P_t, as $\epsilon = -P_t$; where the conjugate momenta is defined as $P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$. $\mathcal{L}$ being the Lagrangian of the string is given by

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)} \tag{40}$$

where $g_{\mu\nu}$ is the target space metric, which describes the embedding of the string in the target space. Therefore the conjugate momentum is then given by

$$P_t = \frac{1}{2\pi\alpha'}\frac{1}{2\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)}}\frac{\partial}{\partial \dot{X}^t}(-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)) \tag{41}$$

To make computation simpler we shall consider individual components of this result and then substitute back in later. Firstly, we find that

$$\begin{aligned} -det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu) &= -det\begin{pmatrix} g_{\mu\nu}\dot{X}^\mu \cdot \dot{X}^\nu & g_{\mu\nu}\dot{X}^\mu \cdot X'^\nu \\ g_{\mu\nu}X'^\mu \cdot \dot{X}^\nu & g_{\mu\nu}X'^\mu \cdot X'^\nu \end{pmatrix} \\ &= -det\begin{pmatrix} g_{tt} + g_{\phi_1\phi_1}\omega^2 & 0 \\ 0 & g_{rr}\left(\frac{dr}{d\sigma}\right) \end{pmatrix} \\ &= \left(f(r) - r^2\frac{\omega^2}{f(r)}\right)\left(\frac{dr}{d\sigma}\right)^2 \end{aligned} \tag{42}$$

Now we wish to compute the last expression in P_0.

$$\frac{\partial}{\partial \dot{X}^t}(-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)) = -\frac{\partial}{\partial \dot{X}^t}\begin{pmatrix} g_{\mu\nu}\dot{X}^\mu \cdot \dot{X}^\nu & g_{\mu\nu}\dot{X}^\mu \cdot X'^\nu \\ g_{\mu\nu}X'^\mu \cdot \dot{X}^\nu & g_{\mu\nu}X'^\mu \cdot X'^\nu \end{pmatrix} \tag{43}$$

But the terms that are not on the diagonal vanish due to the ansatz and we may compute the last term in the determinant as it is not dependent on $\dot{X}^\mu$. Therefore we are left with

$$-\frac{\partial}{\partial \dot{X}^t}\left(g_{\mu\nu}\dot{X}^\mu \cdot \dot{X}^\nu\right)\left(g_{rr}\left(\frac{dr}{d\sigma}\right)^2\right) \tag{44}$$

Using the product rule this expression reduces to

$$-2g_{tt}g_{rr}\left(\frac{dr}{d\sigma}\right)^2 = 2\left(\frac{dr}{d\sigma}\right)^2 \tag{45}$$

Substituting the results of equations (45) and (42) into P_t, then P_t simplifies to

$$P_t = -\frac{1}{2\pi\alpha'}\frac{1}{\sqrt{\left(1 - r^2\frac{\omega^2}{f(r)}\right)}}\left(\frac{dr}{d\sigma}\right) \tag{46}$$

Finally we are now able to determine the energy of a string. Using the fact that $\epsilon = -P_0$ and equation (39), the energy of a string is given by

$$\begin{aligned} E &= \frac{1}{\pi\alpha'}\int_{r_{min}}^{r_{max}} d\sigma\left(\frac{dr}{d\sigma}\right)\frac{1}{\sqrt{\left(1 - r^2\frac{\omega^2}{f(r)}\right)}} \\ &= \frac{1}{\pi\alpha'}\int_{r_{min}}^{r_{max}} dr\frac{1}{\sqrt{\left(1 - r^2\frac{\omega^2}{f(r)}\right)}} \end{aligned} \tag{47}$$

where we have used the fact that $d\sigma(\frac{dr}{d\sigma}) = dr$. Note that since we are integrating from r_{min} to r_{max}, the integral only accounts for one half of the actual string configuration, therefore we have multiplied the integral by 2.

Now that we have obtained an expression of the energy of a string in terms of an integral with boundaries r_{min} and r_{max}, we further simplify the expression by substituting $f(r)$ into (47), then

$$E = \frac{1}{\pi\alpha'}\int_{r_{min}}^{r_{max}} dr\sqrt{\frac{r^4 + r^2 - M}{r^4(1-\omega^2) + r^2 - M}} \tag{48}$$

This expression for the energy of a classical closed string is identical to the one obtained in [4].

5.2 Angular momentum

To compute the angular momentum of the string we use a similar approach as seen when computing the energy of the string. Firstly we determine the angular momentum for a point on the string denoted as s and then integrate over all points of the string in terms of $d\sigma$.

Then the angular momentum of a string is given by

$$S = \int_{r_{min}}^{r_{max}} d\sigma s \tag{49}$$

The angular momentum for a point on the string is given by the conjugate momentum $s = -P_{\phi_1}$. Thus

$$P_{\phi_1} = \frac{1}{2\pi\alpha'} \frac{1}{2\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)}} \frac{\partial}{\partial \dot{X}^{\phi_1}} (-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)) \tag{50}$$

Focusing on the partial derivative of this expression we find that

$$\frac{\partial}{\partial \dot{X}^{\phi_1}} (-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)) = -\frac{\partial}{\partial \dot{X}^{\phi_1}} \begin{pmatrix} g_{\mu\nu}\dot{X}^\mu \cdot \dot{X}^\nu & g_{\mu\nu}\dot{X}^\mu \cdot X'^\nu \\ g_{\mu\nu}X'^\mu \cdot \dot{X}^\nu & g_{\mu\nu}X'^\mu \cdot X'^\nu \end{pmatrix} \tag{51}$$

Due to the ansatz the terms of this determinant that are not on the diagonal vanish. We may compute the last term in the determinant as it is not dependent on $\dot{X}^{\phi_1}$. Therefore this expression becomes

$$-\frac{\partial}{\partial \dot{X}^{\phi_1}} \left(g_{\mu\nu}\dot{X}^\mu \cdot \dot{X}^\nu \right) \left(g_{rr} \left(\frac{dr}{d\sigma} \right)^2 \right) \tag{52}$$

applying the product rule, this reduces to

$$-2\omega g_{\phi_1\phi_1} g_{rr} \left(\frac{dr}{d\sigma} \right)^2 = -2\omega \frac{r^2}{f(r)} \left(\frac{dr}{d\sigma} \right)^2 \tag{53}$$

Substituting this result and (42) into P_{ϕ_1}, we find that

$$P_{\phi_1} = -\frac{1}{2\pi\alpha'} \frac{1}{\sqrt{\left(1 - r^2 \frac{\omega^2}{f(r)}\right)}} \left(\frac{dr}{d\sigma} \right) \tag{54}$$

Therefore the angular momentum of the string is given by

$$\begin{aligned} S &= \frac{1}{\pi\alpha'} \int_{r_{min}}^{r_{max}} d\sigma \left(\frac{dr}{d\sigma} \right) \frac{\omega \frac{r^2}{f(r)}}{\sqrt{\left(1 - r^2 \frac{\omega^2}{f(r)}\right)}} \\ &= \frac{1}{\pi\alpha'} \int_{r_{min}}^{r_{max}} dr \frac{\omega \frac{r^2}{f(r)}}{\sqrt{\left(1 - r^2 \frac{\omega^2}{f(r)}\right)}} \end{aligned} \tag{55}$$

where we have used the fact that $d\sigma(\frac{dr}{d\sigma}) = dr$, and we have multiplied the integral by 2 since we are integrating from r_{min} to r_{max}. As in the energy case we have obtained an expression of the angular momentum of a string in terms of an integral with boundaries r_{min} and r_{max}.
We further simplify this expression by substituting f(r) into equation (55) and we find that

$$S = \frac{1}{\pi\alpha'} \int_{r_{min}}^{r_{max}} dr \frac{\omega r^4}{\sqrt{(r^4 + r^2 - M)(r^4(1-\omega^2) + r^2 - M)}} \tag{56}$$

Which, like the case for the energy, is also identical to the result obtained in [4].

6 Pulsating String

Now that we understand some rough features of the string orbits given by (25), we wish to diversify the types of string configurations we are considering.

We now cast our attention to the case of a pulsating string. A pulsating string is a circular string that is wrapped around the AdS_5 Schwarzschild black hole, which expands and contracts as it moves within the AdS_5 Schwarzschild background [22, 23]. Since the string expands and contracts we would expect to find that $r(\tau, \sigma) = r(\tau)$ [23, 26]. Therefore we propose the case of a pulsating string that is rotating along the ϕ_1 and ϕ_2 directions, with $\dot{\phi}_1 = \omega_1$ and $\dot{\phi}_2 = \omega_2$. Now we consider the ansatz for such a string configuration, which is similar to those ansatz found in [22, 23, 26, 27]. The ansatz is given by

$$X^\mu(\tau, \sigma) = (\tau, r(\tau), \theta(\sigma), \omega_1\tau, \omega_2\tau) \tag{57}$$

For simplicity suppose that $\omega_1 = \omega_2 = \omega$, this will help simplify computations [23]. Now it is possible to determine some rough features of the orbits of a string, we do this by imposing the requirements that the energy and angular momentum expressions for such a string must be real and positive, as seen in [4]. Therefore using similar methods found in section 5, we find that the energy and angular momentum for such a string configuration is given by

$$E = \frac{1}{\pi\alpha'} \int_0^{2\pi} d\theta \frac{r f(r)}{\sqrt{f(r) - \frac{r'^2}{f(r)} - r^2\omega^2}} \tag{58}$$

$$S_{\phi_1} = \frac{1}{\pi\alpha'} \int_0^{2\pi} d\theta \frac{r^3 \omega \sin^2(\theta)}{\sqrt{f(r) - \frac{r'^2}{f(r)} - r^2\omega^2}} \tag{59}$$

$$S_{\phi_2} = \frac{1}{\pi\alpha'} \int_0^{2\pi} d\theta \frac{r^3 \omega \cos^2(\theta)}{\sqrt{f(r) - \frac{r'^2}{f(r)} - r^2\omega^2}} \tag{60}$$

where S_{ϕ_1} and S_{ϕ_2} represent the angular momentum in the ϕ_1 and ϕ_2 directions respectively; and $r' = \frac{dr}{d\tau}$. Substituting the value for $f(r)$ into these equations and evaluating the integrals we find that

$$E = \frac{2}{\alpha'} \frac{(r^4 + r^2 - M)^{\frac{3}{2}}}{\sqrt{(r^4 + r^2 - M)^2 - r^4\omega^2(r^4 + r^2 - M) - r^4 r'}} \tag{61}$$

$$S_\phi = \frac{r^3\omega}{\alpha'} \sqrt{\frac{r^2(r^4 + r^2 - M)}{(r^4 + r^2 - M)^2 - r^4\omega^2(r^4 + r^2 - M) - r^4 r'}} \tag{62}$$

where $S_\phi = S_{\phi_1} = S_{\phi_2}$. From equation (61) we see that the expression within the square root must be strictly positive. Therefore, setting $\alpha = r^4 + r^2 - M$ we are simply left with a second order polynomial with condition

$$\alpha^2 - r^4\omega^2\alpha - r^4 r' > 0 \tag{63}$$

However, α may be negative itself unless $r^2(r^2+1) > M$. Using the quadratic formula we are then able to determine the roots of the polynomial (63), which are

$$\alpha = \frac{r^4\omega^2}{2}\left(1 \pm \sqrt{1 + 4\frac{r'}{\omega^4}}\right) \tag{64}$$

The expression within the square-root must be strictly positive in order to avoid imaginary results, thus we find that we must have

$$\frac{dr}{d\tau} \geq -\frac{\omega^2}{4} \tag{65}$$

A discussion on the physical implications of this requirement is left to the conclusions, section 9.

7 Kerr-AdS$_5$ Black Hole

We now consider the Kerr AdS$_5$ black hole as given in [4]. The Kerr AdS$_5$ black hole is similar to the AdS$_5$ Schwarzschild black hole; however, the Kerr AdS$_5$ black hole introduces a rotational component a.

We shall consider the string configuration as in [4], given by the ansatz (25) in this spacetime. Since the rotation of the string only takes place along one of the circles of the AdS$_5$ black hole it is taken that $\theta = \frac{\pi}{2}$ and $\phi_1 = \phi$, $\phi_2 = 0$ [4]. The resulting metric is given by

$$\begin{aligned} ds^2 &= -\frac{1}{r^2}(\Delta - a^2)dt^2 + \frac{r^2}{\Delta}dr^2 + \frac{1}{r^2(1-a^2)^2}((r^2+a^2)^2 - a^2\Delta)d\phi^2 \\ &\quad + \frac{2a}{r^2(1-a^2)}(\Delta - (r^2+a^2))dtd\phi \end{aligned} \tag{66}$$

where a is the rotation parameter,

$$\Delta = (r^2 + a^2)(1 + r^2) - M \tag{67}$$

and the AdS radius has been scaled to one. We we now wish to determine the turning points r_{min} and r_{max} using a similar method to the AdS_5 Schwarzschild case. Using the Virasoro constraint equation and setting $\frac{dr}{d\sigma} = 0$ we get

$$[(r^2 + a^2)^2 - a^2\Delta]\omega^2 + 4a(1 - a^2)[\Delta - (r^2 + a^2)]\omega - (\Delta - a^2)(1 - a^2)^2 = 0 \tag{68}$$

Manipulating this expression we can show that it gives

$$r^4 + \left(\frac{1 - a^2(\omega + a)^2}{1 - (\omega + a)^2}\right) r^2 - \left(\frac{[1 - a(\omega + a)^2]^2}{1 - (\omega + a)^2}\right)\left(\frac{M}{1 - a^2}\right) = 0 \tag{69}$$

Which is identical to equation (56) in [4] with $V = 0$. Now from this expression we can easily determine the turning points r_{min} and r_{max} using the quadratic formula. Thus

$$\begin{aligned} r^2 &= \frac{1}{2}\left(\frac{a^2(\omega + a)^2 - 1}{(\omega + a)^2 - 1}\right) \\ &\pm \frac{1}{2}\sqrt{\left(\frac{a^2(\omega + a)^2 - 1}{(\omega + a)^2 - 1}\right)^2 + 4\left(\frac{(1 - a(\omega + a)^2)^2}{1 - (\omega + a)^2}\right)\left(\frac{M}{1 - a^2}\right)} \end{aligned} \tag{70}$$

Factorizing out the expression $\left(\frac{a^2(\omega+a)^2-1}{(\omega+a)^2-1}\right)^2$ from the square root and simplifying, we find that r_{min} and r_{max} are given by

$$r^2_{max} = \frac{1 - \omega^2 a^2}{2(\omega^2 - 1)}\left(1 + \sqrt{1 - \frac{4M(\omega^2 - 1)}{(1 - a^2)(1 + a\omega)^2}}\right) \tag{71}$$

$$r^2_{min} = \frac{1 - \omega^2 a^2}{2(\omega^2 - 1)}\left(1 - \sqrt{1 - \frac{4M(\omega^2 - 1)}{(1 - a^2)(1 + a\omega)^2}}\right) \tag{72}$$

From these results we are then able to determine some general characteristics of the orbits. Suppose that $\omega^2 < 1$, then we see that r_{max} will be imaginary if $a^2 < 1$. Therefore, for the condition that r_{max} and r_{min} must be real and positive we require that $\omega^2 > 1$. Furthermore; as $r^2_{max} > 0$ and $r^2_{min} > 0$ must be satisfied, we find that $\omega < \frac{1}{a}$.

These results we find are consistent with those that are found in [4].

8 AdS$_5$-Reissner-Nordström

The Reissner-Nordström black hole can be described as a generalization of the Schwarzschild black hole with an associated electric charge Q [1]. The Reissner-Nordström black hole has no angular momentum and the metric for the AdS$_5$-Reissner-Nordström metric can be obtained from [13, 14].
For the case where $n = 4$ the metric [14] takes the form

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2(\theta)d\phi_1^2 + \cos^2(\theta)d\phi_2^2) \tag{73}$$

with

$$V(r) = 1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + \frac{r^2}{R^2} \tag{74}$$

where R is the characteristic length scale, which we set to $R = 1$; M is related to the ADM mass of the black hole while Q represents the total charge.

We consider the string configuration from [4]. Before we can consider the orbits of such a string configuration we need to understand more about the horizons associated with such a spacetime geometry. The horizons of the AdS-Reissner-Nordström spacetime can be computed from the condition $V(r_\lambda) = 0$, where r_λ is a horizon of the AdS-Reissner-Nordström black hole [15]. In general $V(r)$ has six roots; however, imaginary roots possess no physical implication [13]. Thus we shall only be concerned with the real and positive roots of $V(r)$.

The AdS-Reissner-Nordström black hole contains two horizons, namely the inner and outer horizons [1]. The outer horizon plays the role of the event horizon [14]. Using maple soft 2015 we are able to determine the roots of $V(r)$, at first glance it appears that there is only one positive real root as seen in Appendix B; the first positive real root is given below

$$r_1 = \frac{\sqrt{6\chi^{\frac{1}{3}}\left(\chi^{\frac{2}{3}} + 12M - 2\chi^{\frac{1}{3}} + 4\right)}}{6\chi^{\frac{1}{3}}} \tag{75}$$

where

$$\chi = -36M - 108Q^2 - 8 + 12\sqrt{81Q^4 - 12M^3 + 54MQ^2 - 3M^2 + 12Q^2} \tag{76}$$

From this expression we are able to determine constraints on the variables Q and M. We do this by imposing the requirement that the expression within the square-root of χ be strictly positive. Since we consider the string configuration from [4], we make use of the Virasoro constraint equation (88) and obtain an identical result to equation (26). Which then becomes

$$r^6\left(1 - \omega^2\right) + r^4 - Mr^2 + Q^2 = 0 \tag{77}$$

In order to determine the bounds of this string configuration, namely r_{min} and r_{max} we find the roots of the constraint equation (77). Then we are able to determine the constraints on ω by imposing the requirement that r_{min} and r_{max} be real and positive. Thus we are required to find the positive real roots of this sixth order polynomial (77). We find that the first positive real root is given by

$$r = \sqrt{\frac{-12M\omega^2 + \Omega^{\frac{2}{3}} + 12M + 2\Omega^{\frac{1}{3}} - 4}{6\Omega^{\frac{1}{3}}(\omega^2 - 1)}} \tag{78}$$

where

$$\Omega = 108Q^2\omega^4 + \left(\Upsilon - 216Q^2 - 36M\right)\omega^2 - \Upsilon + 108Q^2 + 36M + 8 \tag{79}$$

and

$$\Upsilon = 12\sqrt{3}\sqrt{27Q^4\omega^4 - \left(54Q^4 + 4M^3 - 18MQ^2\right)\omega^2 + 27Q^4 - M^2(4M+1) + Q^2\left(18M + 4\right)} \tag{80}$$

The exact expressions for r_{min} and r_{max} are not immediately obvious from the roots obtained. Thus future studies should attempt to determine exact expressions for r_{min} and r_{max}. Now, using the root given above we impose the requirement that the term within the square-root of Υ be real and positive, thus we are then able to determine constraints on ω. We find that

$$\omega^2 < 1 + \frac{2M^3 - 9MQ^2 - \zeta}{27Q^4} \tag{81}$$

where

$$\zeta = \sqrt{216M^3Q^4 - 486MQ^6 + 4M^6 - 36M^4Q^2 + 108M^2Q^4 - 27Q^5\left(18M + 4\right)^2} \tag{82}$$

If this requirement is satisfied we find that the closed folded string will orbit around the outer horizon of the black hole. If $Q \to 0$ the term within the square-root of Υ reduces to the constraint (31).

From expression (81) we may conclude that $1 < \omega < \omega_{max}$, where ω_{max} is the upper bound of ω given in (81). If $\omega = \omega_{max}$ then from the example provided in Appendix B we see that the string would then become point-like as there would only be a single positive root for the constraint equation (77). The expression obtained in (81) generalizes the result obtained in [4]; however, this result has not appeared in literature.

9 Conclusions

As seen in section 4, the Virasoro constraint equations are compatible with the geodesic equation for the case of a classical closed string extended radially in the r direction for the AdS_5 Schwarzschild black hole background. The geodesic equation proved to be a nightmare in attempting to obtain the boundaries r_{min} and r_{max} as well as the black hole horizon r_H; however, the Virasoro constraint equations achieved the same task with ease. This is due to the fact that the geodesic equation (23) represents multiple geodesic equations, one for each variable of spacetime; whereas there are only two Virasoro constraint equations. Therefore it is easier to use the Virasoro constraint equations in order to obtain the relevant boundaries r_{min}, r_{max} and r_H. In this case we only needed one as the second Virasoro constraint equation vanished.

We found that one could use various methods when attempting to understand some rough features of string orbits in curved spacetimes. One could use the Virasoro constraint equations while imposing the condition that the boundaries obtained from these equations be real and positive, or we could compute the energy E and angular momentum S of these string orbits while imposing the requirements that E and S be real and positive.

When we considered the pulsating string in section 5, we found the interesting requirement (65). This requirement shows that the angular velocity ω restricts the rate at which the string pulsates.

When we considered a string extended in the radial r direction rotating with angular velocity $\dot{\phi}_1 = \omega$, we found that in the AdS_5 Schwarzschild black hole background the string would orbit around the horizon of the black hole r_H. This was also the case when we considered the Kerr-AdS_5 and AdS_5-Reissner-Nordström Black Hole. We found that the constraint equation for ω was a function of the black hole mass M for the AdS_5 Schwarzschild case; however, for the AdS_5-Reissner-Nordström case the constraint equation for ω was a function of charge Q and mass M of the black hole.

We find that if we set $Q \to 0$, we obtain the result for the AdS_5 Schwarzschild case. Which is to be expected as the AdS_5-Reissner-Nordström black hole is simply a AdS_5 Schwarzschild black hole, but with a small charge Q.

From sections 7 and 8 for the Kerr-AdS_5 and AdS_5-Reissner-Nordström black hole cases respectively, we found the requirement that $\omega > 1$. This implies that the classical closed string must orbit the respective black holes faster than the speed of light. This statement is absurd; however, according to [4] ω must be thought of as an average angular velocity. To resolve this issue we should consider string interactions in future studies, then we would expect to find momentum transfer between the respective black hole and the orbiting string. This momentum transfer would result in the loss of angular momentum of the string, resulting in the string falling behind the event horizon.

Appendix A

S_E is equivalent to the relativistic action

We provide a proof that the action S_E is equivalent to the relativistic action for a point particle. Firstly, we determine the variation of the action S_E with respect to the auxiliary field $e(\tau)$. Thus we are then able to determine the equations of motion for $e(\tau)$.

$$\delta S_E = \frac{1}{2}\int d\tau(-e^{-2}g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} - m^2)\delta e \tag{83}$$

By setting $\frac{\delta S_E}{\delta e} = 0$ we are able to determine the equations of motion for $e(\tau)$. Thus the equations of motion are given by

$$e = \sqrt{\frac{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}}{m^2}} \tag{84}$$

Now we substitute this result into the relativistic action

$$\begin{aligned} S_E &= \frac{1}{2}\int d\tau\left(\left(\sqrt{\frac{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}}{m^2}}\right)^{-1} g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} - m^2\left(\sqrt{\frac{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}}{m^2}}\right)\right) \\ &= \frac{1}{2}\int d\tau m(\sqrt{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}})(-2) \\ &= -m\int d\tau\sqrt{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} \\ &= S_R \end{aligned}$$

Therefore the relativistic action S_R is equivalent to the action S_E.

Polyakov is equivalent to Nambu-Goto

We prove that the Polyakov action is equivalent to the Nambu-Goto action.
We achieve this by eliminating the auxiliary metric $h_{\alpha\beta}$ appearing in the Polyakov action by using the auxiliary metric's equations of motion. In order to obtain the equations of motion for the auxiliary world metric we vary the Polyakov action with respect to $h_{\alpha\beta}$.
The variation of h is given by the following [1]

$$\delta h = -\frac{1}{2}h_{\alpha\beta}\delta h^{\alpha\beta}$$

Now, the variation of the Polyakov action with respect to $h_{\alpha\beta}$ gives

$$\delta S_\sigma = -\frac{1}{2}\int d\sigma d\tau(-\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}h^{\lambda\gamma}g_{\mu\nu}\partial_\lambda X^\mu \cdot \partial_\gamma X^\nu + \sqrt{-h}\delta h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)$$

Setting $\frac{\delta S_\sigma}{\delta h^{\alpha\beta}} = 0$, we find that the equation of motion for the auxiliary field is

$$g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu = \frac{1}{2}h_{\alpha\beta}h^{\lambda\gamma}g_{\mu\nu}\partial_\lambda X^\mu \cdot \partial_\gamma X^\nu$$

taking the square root of the negative determinant of both sides of this equation we obtain

$$\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)} = \frac{1}{2}\sqrt{-h}h^{\lambda\gamma}g_{\mu\nu}\partial_\lambda X^\mu \cdot \partial_\gamma X^\nu$$

Therefore using this result, we consider the Polyakov action and we find that

$$\begin{aligned} S_\sigma &= -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \\ &= -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\frac{1}{2}\sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \\ &= -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{-det(g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu)} \\ &= S_{NG} \end{aligned}$$

Therefore the Polyakov action is equivalent to the Nambu-Goto action. This equivalence however only holds on a classical level.

Virasoro constraint equations

In this section we wish to determine the constraints of a classical closed string in curved spacetimes. According to [1], since the auxiliary metric $h_{\alpha\beta}$ has no kinetic term, the world-sheet energy-momentum tensor $T_{\alpha\beta}$ must vanish. The world-sheet energy-momentum tensor is defined as

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}} \tag{85}$$

where S_σ is the Polyakov action. Recall that for the closed string case the auxiliary metric $h_{\alpha\beta}$ was gauged fixed to $h_{\alpha\beta} = \eta_{\alpha\beta}$. Now the variation of the Polyakov action with respect to the auxiliary metric is easily determined and we find that the world-sheet energy-momentum tensor takes the form

$$T_{\alpha\beta} = -\frac{1}{T}(-\frac{1}{2}h_{\alpha\beta}h^{\lambda\gamma}g_{\mu\nu}\partial_\lambda X^\mu \cdot \partial_\gamma X^\nu + g_{\mu\nu}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu) \tag{86}$$

Due to the gauge choice above and the fact that $T_{\alpha\beta}$ must vanish, this result reduces to

$$T_{00} = \frac{1}{2} g_{\mu\nu}(X'^{\mu} \cdot X'^{\nu} - \dot{X}^{\mu} \cdot \dot{X}^{\nu}) + g_{\mu\nu}\dot{X}^{\mu} \cdot \dot{X}^{\nu} = 0$$

and

$$T_{11} = -\frac{1}{2} g_{\mu\nu}(X'^{\mu} \cdot X'^{\nu} - \dot{X}^{\mu} \cdot \dot{X}^{\nu}) + g_{\mu\nu}X'^{\mu} \cdot X'^{\nu} = 0$$

these equations are simplified to

$$T_{00} = T_{11} = \frac{1}{2}(g_{\mu\nu}X'^{\mu} \cdot X'^{\nu} + g_{\mu\nu}\dot{X}^{\mu} \cdot \dot{X}^{\nu}) = 0 \tag{87}$$

The compacted form of this equation is simply the Virasoro constraint equation given in [12] with $Y^p = 0$, and is given by

$$\dot{X}^{\mu}\dot{X}_{\mu} + X'^{\mu}X'_{\mu} = 0 \tag{88}$$

Using this equation we may now determine the constraint equations for classical closed strings in curved backgrounds.

In a similar fashion we may obtain the second Virasoro constraint equation by considering T_{01}. From [12] the second constraint equation takes the form

$$\dot{X}^{\mu}X'_{\mu} = 0 \tag{89}$$

However due to the string configuration considered in [4] this second Virasoro constraint equation vanishes.

Appendix B.

In this section we provide the code necessary to do computations otherwise deemed tedious and time consuming, but the reader should be familiar with such computations.

Christoffel Symbols.

In order to determine the christoffel symbols for section 4, we use the following code. Firstly we construct the metric

```
> restart :
> with(tensor) :
> coord := [t, r, θ, ϕ1, ϕ2] :
> g_compts := array(symmetric, sparse, 1..5, 1..5) :
```

$$> f := 1 + \frac{r^2}{R^2} - \frac{M}{r^2} :$$

$$> g\_compts_{1,\,1} := -f :$$

$$> g\_compts_{2,\,2} := \frac{1}{f} :$$

$$> g\_compts_{3,\,3} := r^2 :$$

$$> g\_compts_{4,\,4} := r^2 \cdot \sin^2(\theta) :$$

$$> g\_compts_{5,\,5} := r^2 \cdot \cos^2(\theta) :$$

```
> g := create([-1, -1], eval(g_compts))
```

$$g := table\left(\left[compts = \begin{bmatrix} -1 - \frac{r^2}{R^2} + \frac{M}{r^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1 + \frac{r^2}{R^2} - \frac{M}{r^2}} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & 0 & r^2 \cos(\theta)^2 \end{bmatrix},\right.\right. \qquad \textbf{(1.1)}$$

$$index\_char = [-1, -1]\,])$$

Now using the tensor package we can easily determine the Christoffel symbols.

> *tensorsGR*(*coord*, *g*, *contra_metric*, *det_met*, *C1*, *C2*, *Rm*, *Rc*, *R*, *G*, *C*) :

> *displayGR*(*Christoffel2*, *C2*)

The Christoffel Symbols of the Second Kind

non-zero components :

$$\{1,12\} = -\frac{r^4 + M R^2}{r\,(-R^2 r^2 - r^4 + M R^2)}$$

$$\{2,11\} = -\frac{(-R^2 r^2 - r^4 + M R^2)\,(r^4 + M R^2)}{R^4 r^5}$$

$$\{2,22\} = \frac{r^4 + M R^2}{r\,(-R^2 r^2 - r^4 + M R^2)}$$

$$\{2,33\} = \frac{-R^2 r^2 - r^4 + M R^2}{R^2 r}$$

$$\{2,44\} = \frac{(-R^2 r^2 - r^4 + M R^2)\,\sin(\theta)^2}{R^2 r}$$

$$\{2,55\} = \frac{(-R^2 r^2 - r^4 + M R^2)\,\cos(\theta)^2}{R^2 r}$$

$$\{3,23\} = \frac{1}{r}$$

$$\{3,44\} = -\sin(\theta)\,\cos(\theta)$$

$$\{3,55\} = \sin(\theta)\,\cos(\theta)$$

$$\{4,24\} = \frac{1}{r}$$

$$\{4,34\} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\{5,25\} = \frac{1}{r}$$

(1.2)

$$\{5,35\} = -\frac{\sin(\theta)}{\cos(\theta)} \qquad \textbf{(1.2)}$$

AdS Reissner-Nordstrom.

The following code is necessary in order to solve the polynomial equations. Firstly, we consider the polynomial equation concerned with the AdS Reissner-Nordstrom black hole.

```
> restart
> alias(z = RootOf((r^6 + r^4 - M·r^2 + Q^2) = 0, r))
```

$$z \qquad \textbf{(2.1)}$$

```
> allvalues(z) :
```

The output above has been supressed as it is quite lengthy; however, we are only concerned with the real roots of this polynomial. Thus the real roots are given by

```
> allvalues(z)[1]
```

$$\frac{1}{6}\Bigg(\sqrt{6}\,\Bigg(\Big(-36\,M - 108\,Q^2 - 8 + 12\sqrt{81\,Q^4 - 12\,M^3 + 54\,M\,Q^2 - 3\,M^2 + 12\,Q^2}\Big)^{1/3}\Big(\Big(-36\,M - 108\,Q^2 - 8 + 12\sqrt{81\,Q^4 - 12\,M^3 + 54\,M\,Q^2 - 3\,M^2 + 12\,Q^2}\Big)^{2/3} + 12\,M - 2\Big(-36\,M - 108\,Q^2 - 8 + 12\sqrt{81\,Q^4 - 12\,M^3 + 54\,M\,Q^2 - 3\,M^2 + 12\,Q^2}\Big)^{1/3} + 4\Big)\Bigg)^{1/2}\Bigg) \Big/ \Big(-36\,M - 108\,Q^2 - 8 + 12\sqrt{81\,Q^4 - 12\,M^3 + 54\,M\,Q^2 - 3\,M^2 + 12\,Q^2}\Big)^{1/3} \qquad \textbf{(2.2)}$$

```
>
```

The other real root obtained is simply the negative of the root displayed above (2.2). All other roots obtained are imaginary.

Now we compute the roots of the polynomial concerned with the constraint equations.

```
> restart
> alias(z = RootOf((r^6·(1 - ω^2) + r^4 - M·r^2 + Q^2) = 0, r))
```

$$z \qquad \textbf{(2.3)}$$

```
> allvalues(z) :
```

Similarly to the case above the output has been supressed and we find only two real roots, as to be expected. The first real root obtained is given by

```
> allvalues(z)[1]
```

$$\frac{1}{6}\Big(-(6\,\omega^2 - 6)\Big(108\,Q^2\,\omega^4 + 12\sqrt{3}\,\big(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2 + 4\,Q^2\big)^{1/2}\,\omega^2 - 216\,Q^2\,\omega^2 - 36\,M\,\omega^2 - 12\sqrt{3} \qquad \textbf{(2.4)}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2} + 108\,Q^2 + 36\,M + 8)^{1/3}\,(12\,M\,\omega^2 - (108\,Q^2\,\omega^4$$

$$+ 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2}\,\omega^2 - 216\,Q^2\,\omega^2 - 36\,M\,\omega^2$$

$$- 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2} + 108\,Q^2 + 36\,M + 8)^{2/3} - 12\,M - 2\,(108\,Q^2\,\omega^4$$

$$+ 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2}\,\omega^2 - 216\,Q^2\,\omega^2 - 36\,M\,\omega^2$$

$$- 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2} + 108\,Q^2 + 36\,M + 8)^{1/3} - 4))^{1/2} \Big/ ((\omega^2 - 1)\,(108\,Q^2\,\omega^4$$

$$+ 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2}\,\omega^2 - 216\,Q^2\,\omega^2 - 36\,M\,\omega^2$$

$$- 12\,\sqrt{3}$$

$$(27\,Q^4\,\omega^4 - 54\,Q^4\,\omega^2 + 4\,M^3\,\omega^2 - 18\,M\,Q^2\,\omega^2 + 27\,Q^4 - 4\,M^3 + 18\,M\,Q^2 - M^2$$

$$+ 4\,Q^2)^{1/2} + 108\,Q^2 + 36\,M + 8)^{1/3})$$

>

The other real root obtained is simply the negative of the root displayed above (2.2).

We now provide an example to show that there indeed exist string solutions for the AdS Reissner-Nordstrom case. Consider the case where Q=1 and M=4. The horrizons of the black hole are then given by the roots of (74). So

> *restart*

> $with(plots):$

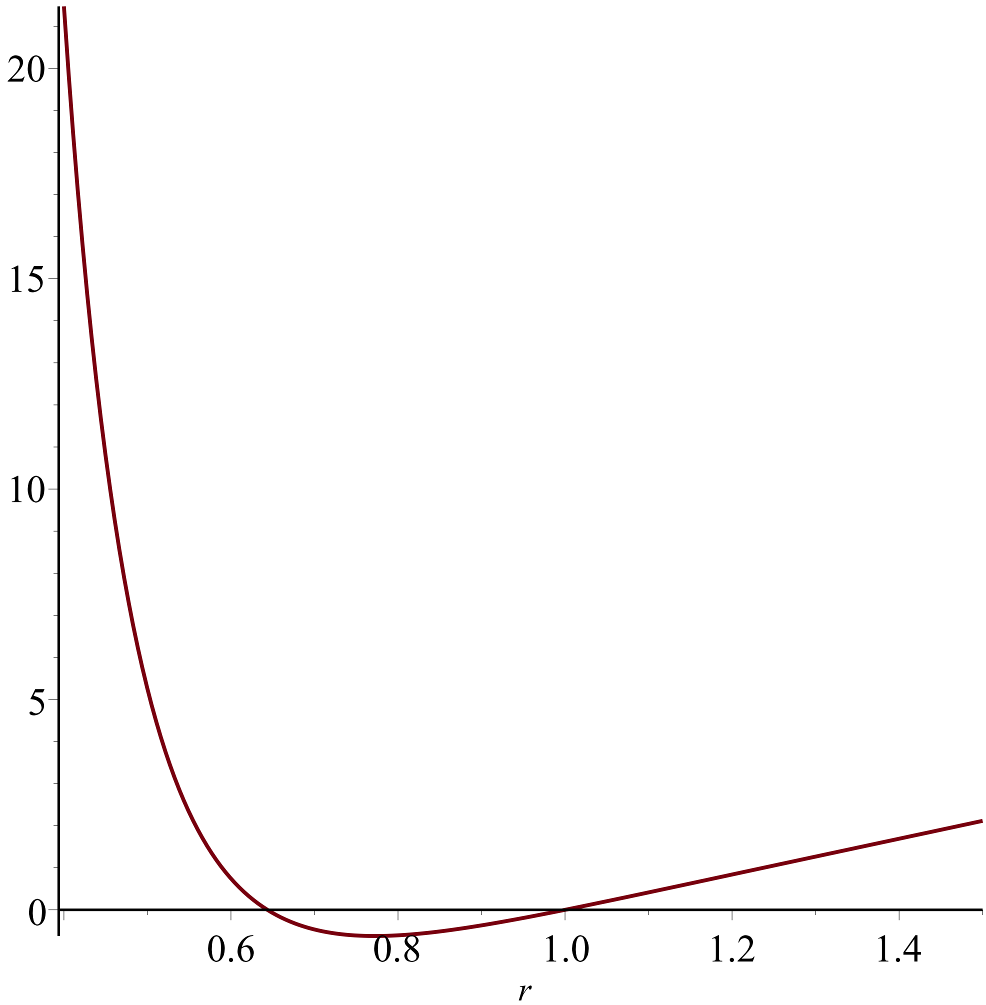
> $Q := 1:$

> $M := 3:$

> $V(r) := 1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + r^2$

$$V := r \rightarrow 1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + r^2 \qquad \textbf{(2.5)}$$

> $plot(V(r), r = 0.4..1.5)$



> $evalf(solve(V(r), r))$

$$1., -1., 0.6435942526, -0.6435942526, 1.553773974\, I, -1.553773974\, I \qquad \textbf{(2.6)}$$

The point r=1 represents the event horizon. Now the bounds of the string are obtained by using the Virasoro constraint equation (88), which gives

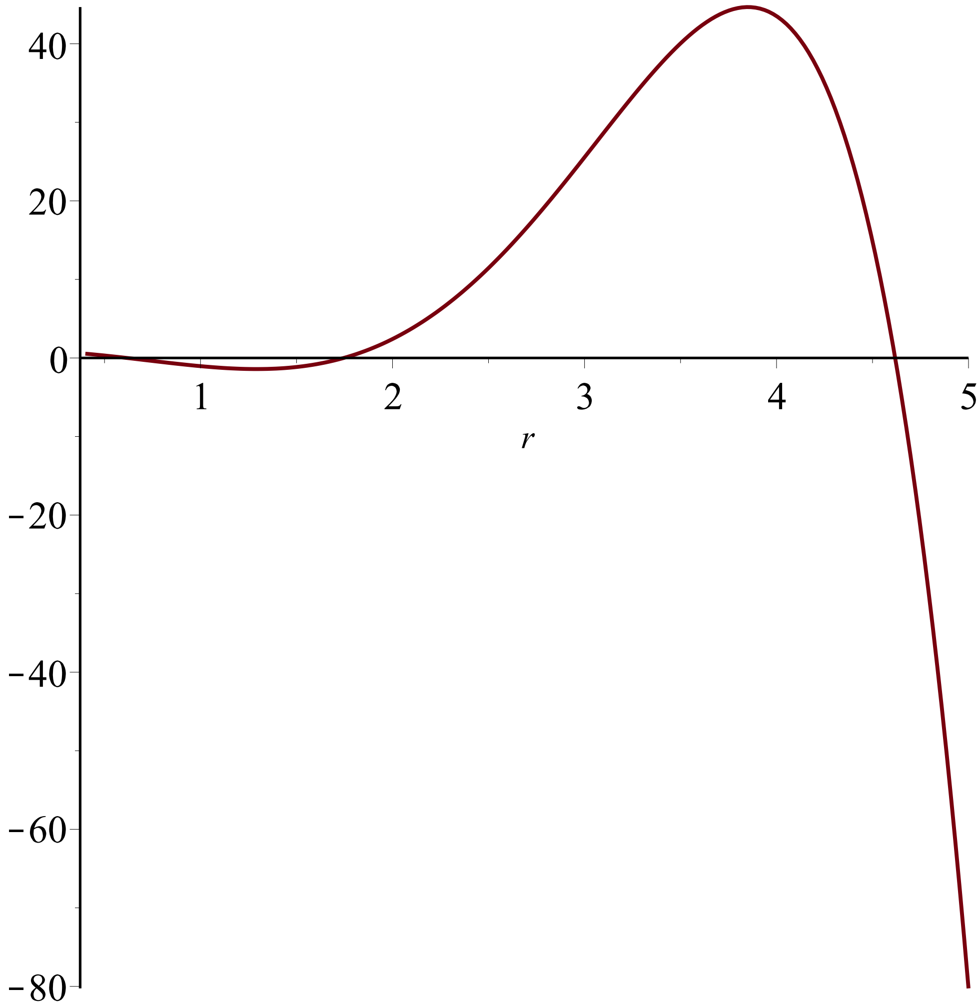
> $drdsigmasquared(r) := r^6 \cdot (1 - \omega^2) + r^4 - M \cdot r^2 + Q^2$

$$drdsigmasquared := r \rightarrow r^6 \left(-\omega^2 + 1\right) + r^4 - M r^2 + Q^2 \qquad \textbf{(2.7)}$$

We choose ω to be close to one since values greater than one are physically absurd. The bounds of the string are given by the roots of the constraint equation (2.7). Thus

> $\omega := 1.02:$

> $plot(drdsigmasquared(r), r = 0.4..5)$



> $evalf(solve(drdsigmasquared(r), r))$

$-0.6172256230, 0.6172256230, -1.745517211, 1.745517211, -4.617865031, 4.617865031$ **(2.8)**

Here we find two positive roots which represent the upper and lower bounds of the string r_min and r_max. We note that these bounds are greater than r=1 which represents the event horizon of the black hole.

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