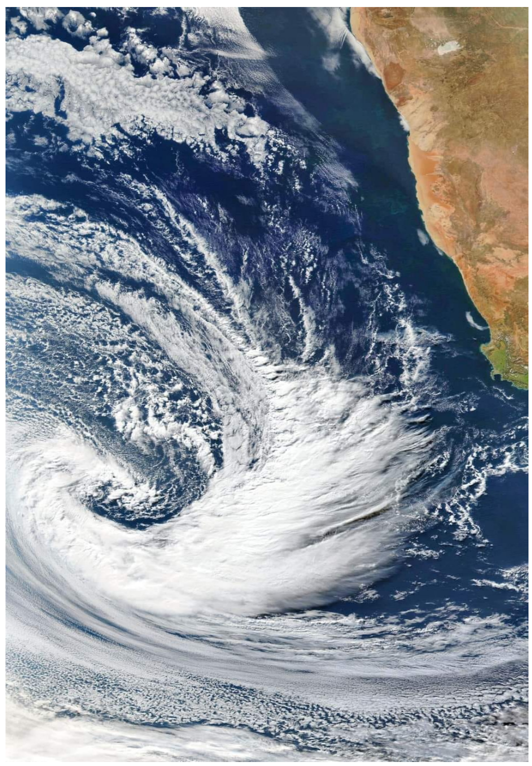
Southern Hemisphere Dynamic Meteorology and its General Circulation



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Is dynamic meteorology for everyone?

Geostrophic adjustment

Synoptic-scale motions in mid-latitudes are in approximate geostrophic balance. In fact, the tendency of winds outside of the tropics, to approach this balance, is a very strong process in the atmosphere. When it happens that the actual winds are not in geostrophic balance with the pressure systems then both the winds and the pressure act together to bring the winds back to geostrophic balance. We now investigate the dynamical adjustment process by which geostrophic balance is achieved. In order to accomplish this objective we consider a continuously stratified atmosphere as a simplified shallow water system. As before we utilise a linear perturbation approach for disturbances about a basic state of no motion with a constant Coriolis parameter. A reasonable value for the Coriolis parameter, f_0, is $-10^{-4}\,\mathrm{s}^{-1}$ at about $45°$S.

Consider the horizontal momentum equation and the continuity equation in vector form

$$\frac{D\vec{V}}{Dt} + f\vec{k}\times\vec{V} = -\vec{\nabla}\Phi, \text{ and}$$
$$\vec{\nabla}\cdot\vec{V} + \frac{\partial\omega}{\partial p} = 0$$

The horizontal momentum equations become

$$\frac{Du}{Dt} - fv = -\frac{\partial\Phi}{\partial x} = -g\frac{\partial z}{\partial x} \simeq -g\frac{\partial h}{\partial x}$$

with h the depth of the layer, and

$$\frac{Dv}{Dt} + fu = -\frac{\partial\Phi}{\partial y} = -g\frac{\partial h}{\partial y}$$

Since the disturbances occur about a basic state of no motion with $f = f_0$

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial h}{\partial x}, \text{ and}$$
$$\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial h}{\partial y}$$

Expanding the variables into their average and deviation from the average values leads to

$$\frac{\partial}{\partial t}(\overline{u} + u') - f_0 v' = -g\frac{\partial}{\partial x}(H + h')$$

The mean meridional flow is zero (mean flow is zonal), and $\overline{u}$ and H are constants

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}, \text{ and}$$
$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

For the continuity equation we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\therefore \int_0^h \frac{\partial w}{\partial z} dz = -\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz$$
$$w(h) - w(0) = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Since $w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t}$

$$\therefore \frac{\partial h}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$
$$\therefore \frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0,$$

because products of perturbations can be neglected.

Take

$$\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial t} - f_0 v' + g\frac{\partial h'}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial v'}{\partial t} + f_0 u' + g\frac{\partial h'}{\partial y}\right] = 0$$
$$\therefore \frac{\partial}{\partial t}\frac{\partial u'}{\partial x} - f_0\frac{\partial v'}{\partial x} + g\frac{\partial^2 h'}{\partial x^2} + \frac{\partial}{\partial t}\frac{\partial v'}{\partial y} - f_0\frac{\partial u'}{\partial y} + g\frac{\partial^2 h'}{\partial y^2} = 0$$
$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) - f_0\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) + g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) = 0$$
$$\therefore \frac{\partial}{\partial t}\left(\frac{1}{H}\frac{\partial h'}{\partial t}\right) - f_0\zeta' + g\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) = 0 \text{ with } \zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$$
$$\therefore \frac{\partial^2 h'}{\partial t^2} - gH\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$

Recall that the shallow water wave speed is $\sqrt{gH}$,

$$\therefore \frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0,$$

with $c^2 = gH$.

Imagine wind field initially in geostrophic balance. Suppose an external process increases the horizontal pressure gradient that leads to a faster geostrophic wind speed. The result is that the pressure gradient force becomes greater than the Coriolis force and the wind is subsequently slightly turned towards low pressure and accelerates the air. This unbalanced flow eventually obtains geostrophic balance again. We next consider the mathematics describing the evolution towards geostrophic balance.

Take

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{\partial v'}{\partial t}+f_0u'+g\frac{\partial h'}{\partial y}\right]-\frac{\partial}{\partial y}\left[\frac{\partial u'}{\partial t}-f_0v'+g\frac{\partial h'}{\partial x}\right]&=0\\
\therefore \frac{\partial}{\partial t}\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)-f_0\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)+g\left(\frac{\partial^2 h'}{\partial x\partial y}-\frac{\partial^2 h'}{\partial x\partial y}\right)&=0\\
\therefore \frac{\partial \zeta'}{\partial t}+f_0\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)&=0\\
\therefore \frac{\partial \zeta'}{\partial t}-\frac{f_0}{H}\frac{\partial h'}{\partial t}&=0\\
\therefore \frac{\partial}{\partial t}\left(\frac{\zeta'}{f_0}-\frac{h'}{H}\right)&=0,
\end{aligned}$$

which implies that $\dfrac{\zeta'}{f_0}-\dfrac{h'}{H}$ is a constant over time.

Consider an idealized shallow water system on a rotating plane, i.e. $f_0\neq 0$, with the following initial conditions:

$$u'=0, v'=0 \text{ and } h'=-h_0\text{sgn}(x),$$

where $\text{sgn}(x)=1$ for $x>0$, and $\text{sgn}(x)=-1$ for $x<0$.

At $x=0$, we have initially h' with the fluid motionless.

Owing to the conservation relationship above, we have

$$\left(\frac{\zeta'}{f_0}-\frac{h'}{H}\right)_{\text{final}}=\left[\left(0-\frac{(-h_0)}{H}\right)\text{sgn}(x)\right]_{\text{initial}}$$

because $\zeta'_{\text{initial}}=0$ with a motionless fluid.

$$\frac{\zeta'}{f_0}-\frac{h'}{H}=\left(\frac{h_0}{H}\right)\text{sgn}(x)$$

$$\begin{aligned}
\therefore \frac{\partial^2 h'}{\partial t^2}-c^2\left(\frac{\partial^2 h'}{\partial x^2}+\frac{\partial^2 h'}{\partial y^2}\right)+f_0H\left[\frac{f_0h'}{H}+\left(\frac{f_0h_0}{H}\right)\text{sgn}(x)\right]&=0\\
\therefore \frac{\partial^2 h'}{\partial t^2}-c^2\left(\frac{\partial^2 h'}{\partial x^2}+\frac{\partial^2 h'}{\partial y^2}\right)+f_0^2h'&=-(f_0^2h_0)\text{sgn}(x)
\end{aligned}$$

h' is initially independent of y, and due to conservation remains so for all time (i.e. $\partial^2h'/\partial t^2=0$). Therefore, the equation reduces to

$$-c^2\frac{d^2h'}{dx^2}+f_0^2h'=-(f_0^2h_0)\text{sgn}(x).$$

Since $h'=h'(x)$, assume the following wave equation for $h'(x)$, namely

$$h'=A\exp(ikx).$$

Given that the amplitude, A, is represented by h_0

$$\frac{d^2}{dx^2}\left(h_0 \exp(ikx)\right) = -k^2 h_0 \exp(ikx) = -k^2 h'$$

$$\therefore -c^2(-k^2 h') + f_0^2 h' = -(f_0^2 h_0)\text{sgn}(x)$$

Consider the homogeneous case of $h_0 = 0$, we get

$$\begin{aligned} c^2 k^2 &= -f_0^2 \\ \therefore k &= \frac{i f_0}{\sqrt{gH}} \text{ since } c^2 = gH \\ &= \frac{i}{\lambda_R} \end{aligned}$$

where $\lambda_R \equiv \dfrac{\sqrt{gH}}{f_0}$ is the Rossby radius of deformation.

$$\begin{aligned} \therefore h' &= h_0 \exp\left(i\left(\frac{i}{\lambda_R}\right)x\right) \\ &= h_0 \exp\left(-\frac{x}{\lambda_R}\right) \end{aligned}$$

Since this solution depends on the sign of x, $h' = h_0 \exp\left(-\dfrac{|x|}{\lambda_R}\right)$.

Consider the previously derived perturbation horizontal momentum equations. Since $h' \neq h'(y)$ and u' and v' are constant over time

$$\begin{aligned} u' &= 0, \text{ and} \\ v' &= \frac{g}{f_0}\frac{\partial}{\partial x}\left(h_0 \exp\left(-\frac{x}{\lambda_R}\right)\right) \\ &= -\frac{g h_0}{f_0 \lambda_R}\exp\left(-\frac{x}{\lambda_R}\right) \end{aligned}$$

Since this solution also depends on the sign of x,

$$v' = -\frac{g h_0}{f_0 \lambda_R}\exp\left(-\frac{|x|}{\lambda_R}\right).$$

Because $\dfrac{\zeta'}{f_0} - \dfrac{h'}{H}$ (the conservation relationship) is constant over time we are able to determine the steady-state geostrophically adjusted velocity and height fields without performing an integration over time. However, if the evolution of the adjustment process is needed, we need to solve for

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0^2 h' = -(f_0^2 h_0)\text{sgn}(x)$$

subject to the initial conditions specified above.

To further investigate the evolution process, consider a simplified version of the horizontal momentum equations and the continuity equations for the case of no y dependence:

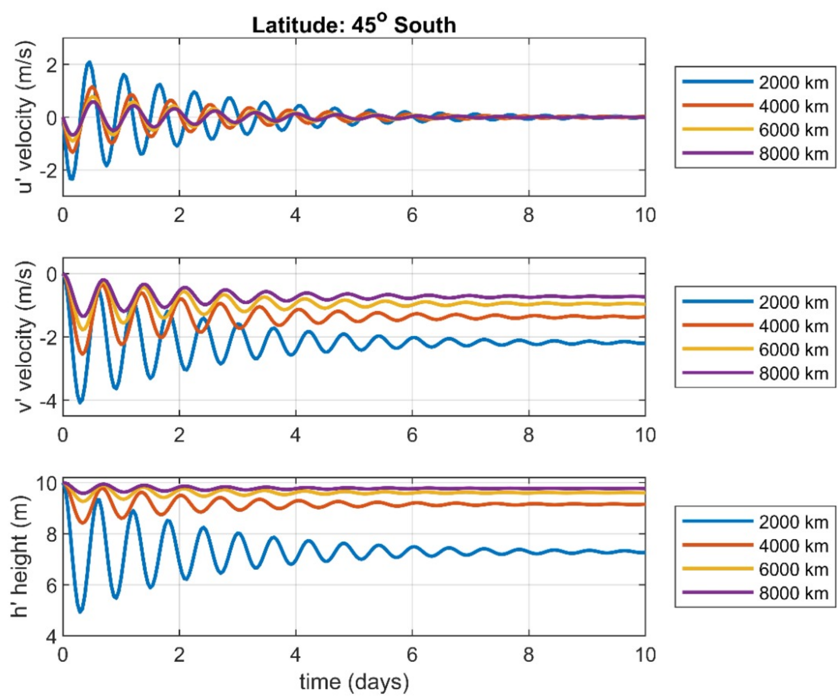
$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$
$$\frac{\partial v'}{\partial t} + f_0 u' = 0$$
$$\frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} = 0$$

Such differential equations can be solved using intrinsic functions of a programming language with the following initial conditions

$u' = 0$
$v' = 0,$ and
h' represented by a sinusoidal function.

Since $h' = \dfrac{\Phi'}{g}$, the solution is for a sinusoidally varying height field.

The figure below shows the results for latitude $45°$S for various wavelengths, and for a time evolution of 10 days. Take note that the final balanced wind only has a meridional (v') component, i.e. all the u' velocities stabilizes at a zero value, and that the geostrophically adjusted heights' stability is dependent on wavelength.



We have already explained in simple terms how geostrophic balance is reached when an external process acts on the horizontal pressure gradient. Another very important example where geostrophic adjustment plays a role is in the initialization process of numerical weather models. If the initial conditions are not in quasi-geostrophic balance, an unbalanced portion of the initial field will project onto inertia-gravity waves. These waves disperse quite fast, and after a while the amplitude of the inertia-gravity waves becomes much smaller, so that all that remains are fields in quasi-geostrophic balance.

Equatorial Kelvin waves

Equatorial waves are eastward as well as westward propagating disturbances, in both the ocean and atmosphere, that are "trapped" about the equator, acting as a waveguide. Equatorial Kelvin waves, which are waves that move only towards the east, play an important role in the dynamics of ENSO by transferring changes in conditions in the western Pacific to the eastern Pacific. For example, tropical convection can give rise to atmospheric equatorial waves, which in turn can cause the effects of convection to be transported over large longitudinal distances.

Since we are interested only in the tropics for this discussion, we consider an equatorial β-plane. In such a case, $\beta \equiv 2\Omega/a$ with Ω the angular velocity of the Earth ($7.292 \times 10^{-5}\,\mathrm{s}^{-1}$), a the radius of the Earth ($6.37 \times 10^6\,\mathrm{m}$), and the Coriolis parameter f is approximated by βy, with y the distance from the equator. The approximation $f \approx \beta y$ ($\therefore\ y \simeq f/\beta$) is mainly restricted to latitudes of 30° north and south of the equator, which is just beyond $3000\,\mathrm{km}$ north and south of the equator. For 20° of latitude, the distance is about $2000\,\mathrm{km}$.

In order to obtain an understanding of the dynamics of Kelvin waves, consider this set of equations that represent the horizontal momentum equations and the continuity equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y}, \text{ and} \\
\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0,
\end{aligned}
$$

with h ($= h(x, y, t)$) the thickness of a layer of fluid, and H the mean depth of the layer.

Because Kelvin waves tend to get trapped in a zone around the equator and move only towards the east, we can simplify the above equations by assuming that v is small enough to ignore entirely. The equations then become

$$
\begin{aligned}
\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} &= 0, \\
fu + g\frac{\partial h}{\partial y} &= 0, \text{ and} \\
\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} &= 0.
\end{aligned}
$$

From the first equation we get $\dfrac{\partial u}{\partial t} = -g\dfrac{\partial h}{\partial x}$. For the remaining two equations we differentiate both with respect to t:

$$\frac{\partial}{\partial t}\left(fu + g\frac{\partial h}{\partial y}\right) = 0$$
$$\therefore f\frac{\partial u}{\partial t} + g\frac{\partial^2 h}{\partial t \partial y} = 0$$
$$\therefore f\left(-g\frac{\partial h}{\partial x}\right) + g\frac{\partial^2 h}{\partial t \partial y} = 0$$
$$\therefore \frac{\partial^2 h}{\partial t \partial y} - f\frac{\partial h}{\partial x} = 0$$
$$\text{Also, } \frac{\partial}{\partial t}\left(\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x}\right) = 0$$
$$\therefore \frac{\partial^2 h}{\partial t^2} + H\frac{\partial}{\partial x}\left(-g\frac{\partial h}{\partial x}\right) = 0$$
$$\therefore \frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

Equatorial Kelvin waves as travelling waves, propagte in the zonal direction and decay in the meridional direction. We therefore consider an elementary solution of the form

$$h(x, y, t) = \gamma(y)e^{i(kx - \nu t)},$$

where k is the zonal wave number, ν the frequency, and $\gamma(y)$ decaying as $|y| \to \infty$. Take note that the phase speed, c, of the travelling waves is related to the frequency so that

$$\nu = \frac{c}{k}.$$

Next, we apply the operators of the partial differential equations for h:

$$\frac{\partial^2 h}{\partial t \partial y} = -i\nu\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial x} = ikh$$
$$\frac{\partial^2 h}{\partial t^2} = -\nu^2 h$$
$$\frac{\partial^2 h}{\partial x^2} = -k^2 h$$

$$\therefore -\nu^2 h - gH(-k^2 h) = 0$$
$$\therefore \nu^2 = gHk^2$$

Also,

$$\therefore -i\nu\frac{\partial h}{\partial y} - f(ikh) = 0$$
$$\therefore -i\nu\left(\frac{\partial \gamma}{\partial y}e^{i(kx-\nu t)} + \gamma \times 0\right) - \beta yikh = 0$$

Multiplying throughout by γ:

$$\therefore -i\nu\frac{\partial\gamma}{\partial y}\left(\gamma e^{i(kx-\nu t)}\right) - \gamma\beta yikh = 0$$

$$\therefore -i\nu\frac{\partial\gamma}{\partial y} - ik\gamma\beta y = 0$$

$$\therefore \frac{\partial\gamma}{\partial y} = \frac{d\gamma}{dy} = -\frac{\beta k}{\nu}y\gamma(y)$$

Since $\beta > 0$ and $k > 0$, and if $\nu > 0$, it follows that $\dfrac{d\gamma}{dy} < 0$, which is in agreement with the assumption that $\gamma(y)$ decays as $|y| \to \infty$.

A solution for $\gamma(y) = e^{-\frac{\beta k}{2\nu}y^2}$, because

$$\begin{aligned}\frac{d\gamma}{dy} &= -\frac{2\beta k}{2\nu}ye^{-\frac{\beta k}{2\nu}y^2} \\ &= -\frac{\beta k}{\nu}y\gamma(y).\end{aligned}$$

Therefore, in order for $\gamma(y)$ to decay, $\nu > 0$. However,

$$\nu^2 = gHk^2$$

$$\therefore \nu = +\sqrt{gH}k$$

Since $c = \nu/k$, and the wave group speed, $c_g = \partial\nu/\partial k$

$$c = +\sqrt{gH} \text{ and } c_g = +\sqrt{gH}$$

Because $c = c_g$, equatorial Kelvin waves propagate without dispersion. When waves are dispersive, their phase speed varies with k, and the various wave components of a disturbance that originated at a given location are found in different places at a later time. For non-dispersive Kelvin waves, a group of such waves will preserve its shape as it propagates in space at the phase speed of the wave ($c_g = c$).

We can now finalize the equation that describes the thickness of the layer of a fluid:

$$\begin{aligned}h(x,y,t) &= \gamma(y)e^{i(kx-\nu t)} \\ &= e^{-\left(\frac{\beta k}{2\nu}\right)y^2}e^{i(kx-\nu t)} \\ &= e^{-\left(\frac{\beta}{2c}\right)y^2}e^{ik(x-ct)}\end{aligned}$$

Because of the importance of Kelvin wave sin ENSO dynamics, we discuss the application of the above equation in $h(x,y,t)$ as a component of ocean waves. In fact, h is considered here to be the layer thickness variation (thickening counted positively and thinning counted negatively) over time.

Consider the dispersion of a perturbation generated by a prolonged (i.e., more than a week) wind anomaly imposed on a stretch of equatorial ocean. The figure below shows the results from a finite difference method for the equatorial shallow water model that uses the leapfrog numerical scheme. Only the first 10 days of the integration are shown. Shallow water waves can also occur at interfaces within the ocean where there is a very sharp temperature gradient called the thermocline. The figure displays the temporal dispersion of a thermocline displacement that came about because of the said wind anomaly. One can clearly see the one-bulge equatorial Kelvin wave propagating eastward, and the much stronger double-bulge Rossby wave propagating westward. Next we evaluate mathematically when such an eastward propagating wave encounters an eastern boundary such as the west coast of South America.

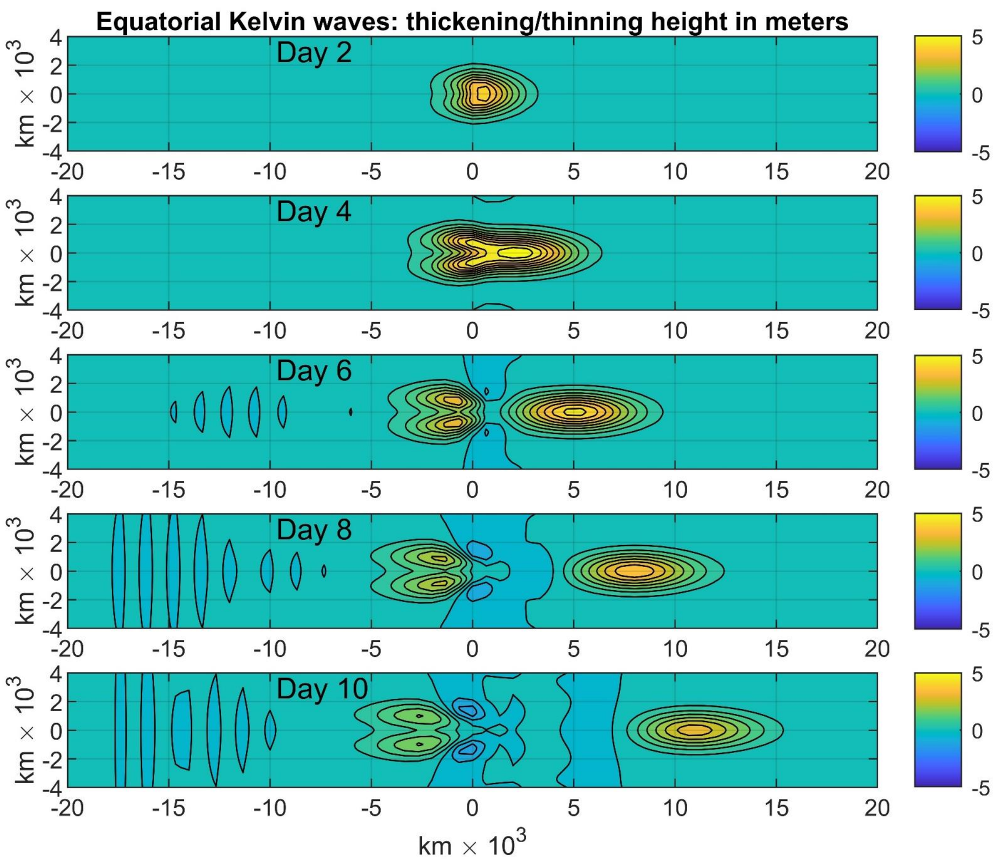


Figure 1: Numerical evolution of the height field on an equatorial beta plane at the times indicated. The Kelvin wave moves eastward and the off equatorial Rossby waves move more slowly westward.

Upon making impact on the South American west coast, the equatorial Kelvin waves split up and the resulting coastal Kelvin waves propagate poleward into the Northern and Southern Hemisphere. For these coastal Kelvin waves we ignore zonal wind speeds $(u = 0)$ and assume a constant Coriolis parameter, f_0. This configuration means that the momentum and continuity equations become

$$-f_0 v + g\frac{\partial h_c}{\partial x} = 0, \text{ with } h_c \text{ the thickness of the layer along the coast}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h_c}{\partial y} = 0, \text{ resulting in } \frac{\partial v}{\partial t} = -g\frac{\partial h_c}{\partial y},$$

$$\text{and } \frac{\partial h_c}{\partial t} + H\frac{\partial v}{\partial y} = 0.$$

Differentiating with respect to t:

$$\frac{\partial}{\partial t}\left(\frac{\partial h_c}{\partial t} + H\frac{\partial v}{\partial y}\right) = 0$$
$$\therefore \frac{\partial^2 h_c}{\partial t^2} + H\frac{\partial}{\partial y}\left(-g\frac{\partial h_c}{\partial y}\right) = 0$$
$$\therefore \frac{\partial^2 h_c}{\partial t^2} - gH\frac{\partial^2 h_c}{\partial y^2} = 0$$

For the coastal Kelvin waves that propagate only in the meridional direction, we consider a solution of the form

$$h_c(x,y,t) = \gamma_\ell(x)e^{i(\ell y - \nu t)}$$

Applying operators:

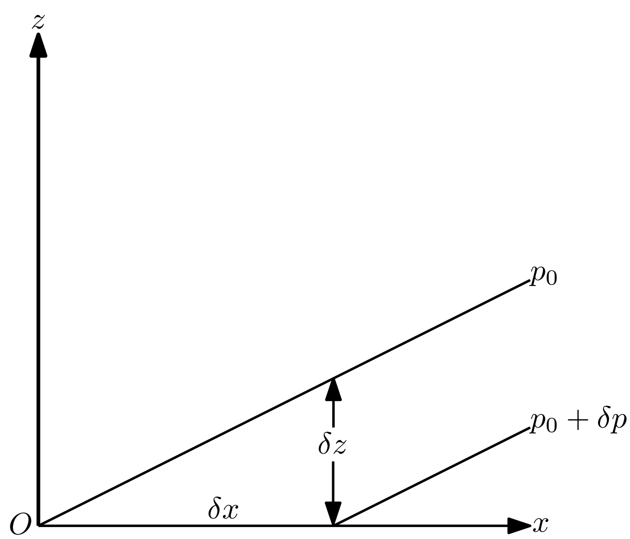
$$\frac{\partial^2 h_c}{\partial t^2} = (-i\nu)^2 h_c = -\nu^2 h_c$$
$$\frac{\partial^2 h_c}{\partial y^2} = (i\ell)^2 h_c = -\ell^2 h_c$$

$$\therefore -\nu^2 h_c - gH(-\ell^2 h_c) = 0$$
$$\therefore \nu^2 = gH\ell^2$$
$$\therefore \nu = \pm\sqrt{gH}\ell$$
$$\text{Since } c = \frac{\nu}{\ell} = \pm\sqrt{gH}$$
$$\text{and } c_g = \frac{\partial \nu}{\partial \ell} = \pm\sqrt{gH}$$

Therefore, coastal Kelvin waves are also non-dispersive because $c = c_g$.

Pressure as a vertical coordinate

Transformation of the horizontal pressure **gradient** force from height to pressure coordinates:



$$\left[\frac{(p_0+\delta p)-p_0}{\delta x}\right]_z = \left[\frac{(p_0+\delta p)-p_0}{\delta z}\right]_x \left(\frac{\delta z}{\delta x}\right)_p$$

Limit as $\delta z \to 0$: $\left[\frac{(p_0+\delta p)-p_0}{\delta z}\right]_x \to -\left(\frac{\partial p}{\partial z}\right)_x$

Limit as $\delta x \to 0$: $\left[\frac{(p_0+\delta p)-p_0}{\delta x}\right]_z \to \left(\frac{\partial p}{\partial x}\right)_z$

$$\therefore \left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

Hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g$

$$\therefore \left(\frac{\partial p}{\partial x}\right)_z = -(-\rho g)\left(\frac{\partial z}{\partial x}\right)_p$$

$$\therefore -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p$$

Recall the geopotential

$$\Phi = \int_0^z g dz \implies \partial\Phi = g\partial z$$

$$\therefore -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$

Isobaric: Characterised by equal or constant pressure, with respect to either space or time.

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p \tag{1.25}$$

and

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p \tag{1.26}$$

$\implies$ In the isobaric coordinate system, the horizontal pressure gradient force is measured by the gradient of **geopotential** at constant pressure.

Advantage of isobaric system: density is not explicit in the pressure gradient force.

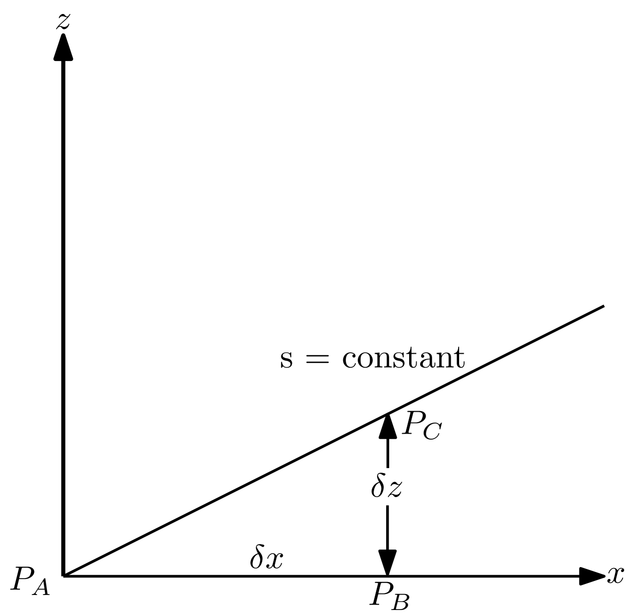
Note:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z - \frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\frac{1}{\rho}\vec{\nabla}_z p$$

$$-\left(\frac{\partial \Phi}{\partial x}\right)_p - \left(\frac{\partial \Phi}{\partial y}\right)_p = -\vec{\nabla}_p \Phi$$

A generalized vertical coordinate

Aim: To obtain a general expression for the horizontal pressure gradient; which is applicable to any vertical coordinate $s = s(x, y, z, t)$



Gradient: $\frac{P_C - P_B}{\delta x}$

$$\frac{P_C - P_B}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x}$$

$$\frac{P_C - P_B}{\delta x} - \frac{P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_B - P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_A}{\delta x} - \frac{P_B}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} + \frac{P_B - P_A}{\delta x}$$

$$\begin{aligned} P_C - P_A &: \text{along diagonal where } s \text{ is constant} \\ \frac{\delta z}{\delta x} &: \text{its diagonal is constant } s \\ P_B - P_A &: \text{along } x \text{ where } z \text{ is constant} \end{aligned}$$

Taking the limits as δx, $\delta z \to 0$

$$\implies \left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \tag{1.27}$$

Identity: $\frac{\partial p}{\partial z} = \left(\frac{\partial s}{\partial z}\right)\left(\frac{\partial p}{\partial s}\right)$

$$\begin{aligned} \therefore \left(\frac{\partial p}{\partial x}\right)_s &= \left(\frac{\partial s}{\partial z}\right)\left(\frac{\partial p}{\partial s}\right)\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \\ &= \left(\frac{\partial p}{\partial x}\right)_z + \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s\left(\frac{\partial p}{\partial s}\right) \end{aligned} \tag{1.28}$$

Basic equations in isobaric coordinates

The horizontal momentum equation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (2.24)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2.25)$$

In vertical form:

$$\frac{D\vec{V}}{Dt} + f\vec{k}\times\vec{V} = -\frac{1}{\rho}\vec{\nabla}p \quad (3.1)$$

where $\vec{V} = \vec{i}u + \vec{j}v$ is the horizontal velocity.

From page 12:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p \text{ and } -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p$$

$$\frac{D\vec{V}}{Dt} + f\vec{k}\times\vec{V} = -\vec{\nabla}_p\Phi \qquad (\delta p < 0)$$

$\vec{\nabla}_p$: **horizontal** gradient operator (p held constant)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p}$$
$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

where $\omega = \dfrac{Dp}{Dt}$ is the "omega" vertical motion, the pressure change following the motion.

Consider equation (2.24) again:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

with fv the Coriolis force and $\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ the pressure gradient force (Pgf).

On the synoptic scale, the order of magnitude of the meridional wind speed, v, is about $10\,\mathrm{m\,s^{-1}}$. Therefore, the order of magnitude of the Coriolis force, $|fv|$, is $\left(10^{-4}\,\mathrm{s^{-1}}\right)\left(10\,\mathrm{m\,s^{-1}}\right)$, which is $10^{-3}\,\mathrm{m\,s^{-2}}$.

Also on the synoptic scale, the Pgf, $\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ has a scale of approximately

$$\begin{aligned}\frac{1}{1\,\mathrm{kg\,m^{-3}}}\frac{10\,\mathrm{hPa}}{1000\,\mathrm{km}} &= \frac{\left(\mathrm{kg^{-1}\,m^{3}}\right)\left(1000\,\mathrm{N\,m^{-2}}\right)}{1\ 000\ 000\,\mathrm{m}}\\ &= 10^{-3}\,\mathrm{kg^{-1}\,kg\,m\,s^{-2}}\\ &= 10^{-3}\,\mathrm{m\,s^{-2}}\end{aligned}$$

The Coriolis force and Pgf are, therefore, of similar scales. We can thus infer that $\dfrac{Du}{Dt}$ (acceleration) must be small, provided that significant flow curvature is absent, in order for the two forces to be in balance.

Geostrophic wind

$\vec{V}_g = \vec{i}u_g + \vec{j}v_g$

Vectorial form of the geostrophic wind:

$$\vec{V}_g \equiv \vec{k} \times \frac{1}{\rho f}\vec{\nabla} p$$

$$\begin{aligned}\therefore f\vec{V}_g &= \vec{k} \times \frac{1}{\rho}\vec{\nabla} p \qquad \left(-\frac{1}{\rho}\delta p = g\delta z = \delta\Phi\right)\\ &= \underbrace{\vec{k} \times \vec{\nabla}_p \Phi}_{\text{No density term!}}\end{aligned} \tag{3.4}$$

Thus, a given **geopotential gradient** implies the same geostrophic wind at any height, whereas a given **horizontal pressure gradient** implies different geostrophic wind values depending on the density.

From the vector form of the geostrophic wind and the definition of ageostrophic wind:

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \text{ and } v_a = v - v_g$$

$$\begin{aligned}\therefore v_a &= v - \frac{1}{\rho f}\frac{\partial p}{\partial x}\\ \therefore f v_a &= f v - \frac{1}{\rho}\frac{\partial p}{\partial x}\end{aligned}$$

Ageostrophic flow must therefore also be small on the synoptic scale.

As a result of the approximate balance between the Coriolis force and Pgf, both acceleration and ageostrophic motion are small on the synoptic scale.

Additionally, it can be shown that for constant Coriolis parameter ($f = f_0$) that the geostrophic wind ($\vec{V}_g$) is non divergent. This implies that the flow is purely horizontal (i.e., $\omega = 0$).

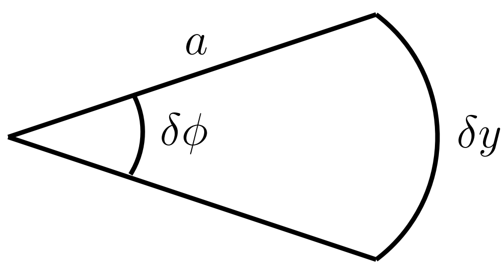
Exercise: The definition of the geostrophic wind in vector form is

$$f\vec{V}_g = \vec{k} \times \vec{\nabla}_p \Phi$$

Derive the divergence of the geostrophic wind for BOTH a constant and variable definition of the Coriolis parameter. For a variable Coriolis parameter, first show that

$$\vec{\nabla} \cdot \vec{V}_g = -\frac{\beta}{f} v_g,$$

then consider



where $a\ (= R_E)$ is the radius of the Earth, ϕ is the latitude and y the length along a latitude circle, to show that for a variable Coriolis parameter the divergence of geostrophic wind is equal to

$$-v_g \frac{\cot\phi}{R_E}$$

Solution:

For constant Coriolis parameter:

$$\begin{aligned}
f\vec{V}_g &= \vec{k} \times \vec{\nabla}\Phi \\
\vec{\nabla} \cdot (f\vec{V}_g) &= \vec{\nabla} \cdot \left(\vec{k} \times \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \right) \Phi \right) \\
f\vec{\nabla} \cdot \vec{V}_g &= \vec{\nabla} \cdot \left(\frac{\partial}{\partial x}\vec{j} - \frac{\partial}{\partial y}\vec{i} \right) \Phi \\
&= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \right) \cdot \left(-\frac{\partial}{\partial y}\vec{i} + \frac{\partial}{\partial x}\vec{j} \right) \Phi \\
&= \left(-\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y \partial x} \right) \Phi \\
\therefore \vec{\nabla} \cdot \vec{V}_g &= 0
\end{aligned}$$

For variable Coriolis parameter:

We have shown above that $\vec{\nabla} \cdot \left[\vec{k} \times \vec{\nabla}\Phi\right] = 0$

$$
\begin{aligned}
\vec{\nabla} \cdot (f\vec{V}_g) &= 0 \\
\therefore \vec{\nabla} \cdot (fu_g\vec{i} + fv_g\vec{j}) &= 0 \\
\therefore \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right) \cdot (fu_g\vec{i} + fv_g\vec{j}) &= 0 \\
\therefore \frac{\partial}{\partial x}(fu_g) + \frac{\partial}{\partial y}(fv_g) &= 0 \\
\therefore \frac{\partial f}{\partial x}u_g + f\frac{\partial u_g}{\partial x} + \frac{\partial f}{\partial y}v_g + f\frac{\partial v_g}{\partial y} &= 0 \\
\therefore f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) &= -\frac{\partial f}{\partial y}v_g \\
\therefore f\vec{\nabla} \cdot \vec{V}_g &= -\beta v_g \\
\therefore \vec{\nabla} \cdot \vec{V}_g &= -\frac{\beta}{f}v_g \qquad \text{(A)}
\end{aligned}
$$

From the figure

$$
\begin{aligned}
\delta y &= a\delta\phi \\
\therefore \frac{1}{\delta y} &= \frac{1}{a}\frac{1}{\delta\phi}
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \frac{\partial f}{\partial y} \\
&= \frac{1}{a}\frac{\partial f}{\partial\phi} \\
&= \frac{1}{a}\frac{\partial}{\partial\phi}(2\Omega\sin\phi) \\
&= \frac{2\Omega}{a}\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
\therefore \vec{\nabla} \cdot \vec{V}_g &= -\frac{2\Omega}{a}\cos\phi(2\Omega\sin\phi)^{-1}v_g \\
&= -\frac{v_g}{a}\frac{\cos\phi}{\sin\phi} \\
&= -v_g\frac{\cot\phi}{a} \\
&= -v_g\frac{\cot\phi}{R_E} \qquad \text{(B)}
\end{aligned}
$$

Equation (A) can be analysed in order to show the variability of divergence with latitude. The divergence of $\vec{V}_g$ is directly proportional to β and indirectly proportional to f. Figure 2 shows the variability of the Coriolis parameter, f, with latitude. Many features of f become immediately apparent. Firstly, as we know, the Coriolis parameter is negative in the SH and positive in the NH. The magnitude of the Coriolis parameter reaches a maximum at the poles and is zero at the equator. Values of the Coriolis parameter are of the order $10^{-4}\,\mathrm{s}^{-1}$. From Equation (A), it can be seen that the equation is undefined at the equator where $f = 0$. However as $|f| \rightarrow 0$ close to the equator, it follows from equation (A) that the divergence becomes large and $\left|\vec{\nabla} \cdot \vec{V}_g\right| \rightarrow \infty$. Conversely, as $|f|$ increases towards its maximum value, the divergence of $\vec{V}_g$ becomes smaller such that $\left|\vec{\nabla} \cdot \vec{V}_g\right| \rightarrow 0$. For completeness, the β-term is also shown in Figure 3. The magnitude of the β-term increases to a maximum towards the equator and decreases towards the poles. Note that β is very small and of the order of $10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$. The contribution of β to the amount of divergence is therefore similar to that of f where it contributes to a greater magnitude of divergence towards the equator whilst a lesser magnitude of divergence close to the poles.

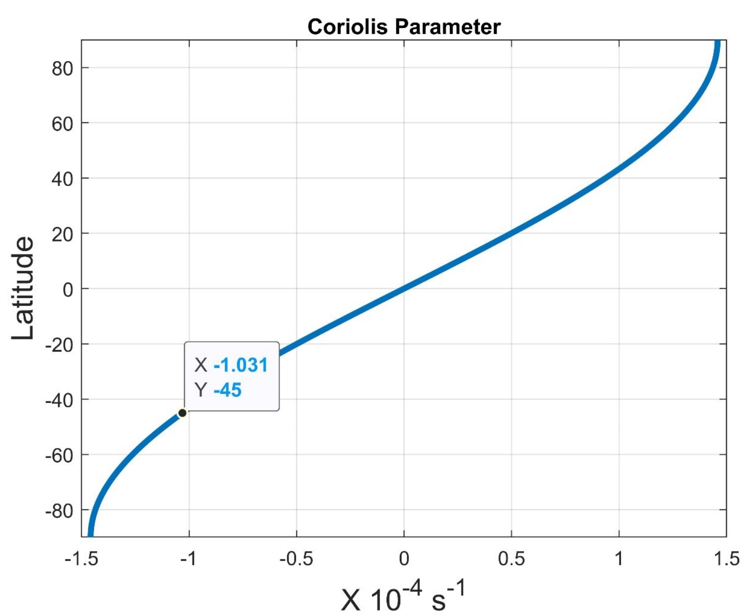


Figure 2: The variability of the Coriolis parameter with latitude.

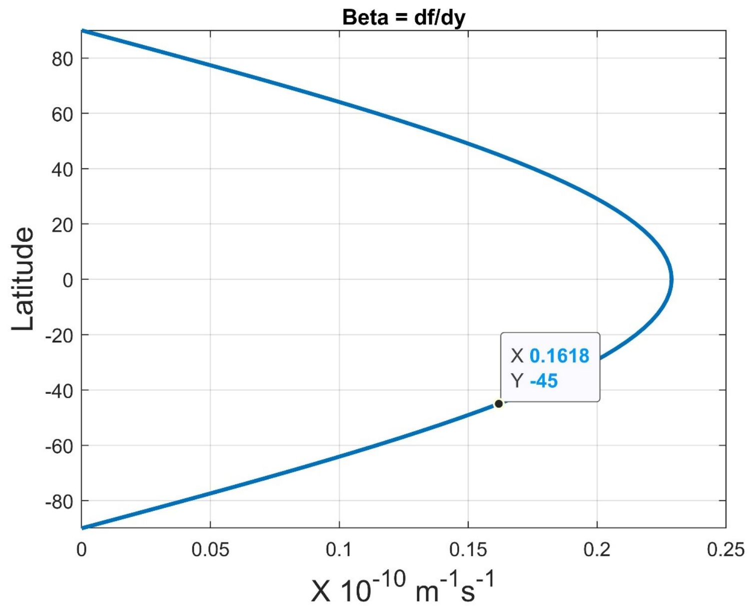


Figure 3: The variation of beta with latitude. Take note that the value of β is of the order $10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$ at $45°$S.

A calculation of the divergence term in Equation (B) provides corroboration of our analysis of Equation (A). The variability of divergence as given by Equation (A) is plotted in Figure 4 below for a number of different v_g values.

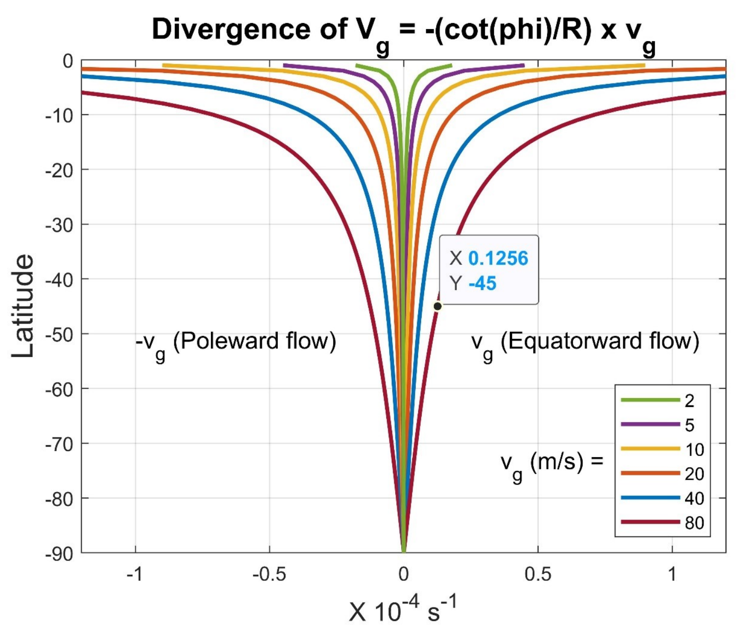


Figure 4: The divergence of the geostrophic wind for a Coriolis parameter that is not constant.

As shown mathematically above, Figure 4 shows that for any v_g the divergence of $\vec{V}_g \to \infty$ as the latitude $\to 0$. Conversely, the divergence of $\vec{V}_g \to 0$ as latitude $\to -90°$. For equatorward flow ($v_g > 0$), divergence occurs, whilst for poleward flow ($v_g < 0$), convergence (negative divergence) occurs.

Importantly, the amount of divergence is small for the majority of the hemisphere, with the exception of flow near the equator. Near the equator, the Coriolis force is very weak or close to zero and thus the geostrophic flow is divergent (i.e., the geostrophic approximation does not hold here since the Coriolis and pressure gradient forces are not in balance). The lack of Coriolis force at the equator is the primary reason why Tropical Cyclones cannot form close to the equator.

The continuity equation

Lagrangian control volume: $\delta V = \delta x \delta y \delta z$

Hydrostatic equation:

$$\begin{aligned}\frac{\delta p}{\delta z} &= -\rho g \\ \therefore \delta z &= -\frac{1}{\rho g}\delta p \\ \therefore \delta V &= -\frac{1}{\rho g}\delta x \delta y \delta p\end{aligned}$$

Mass, conserved following the motion:

$$\begin{aligned}\delta M &= \rho \delta V \\ \therefore \delta M &= -\frac{1}{g}\delta x \delta y \delta p\end{aligned}$$

Thus,

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

The last expression follows from the conservation of mass where $\frac{D}{Dt}(\delta M) = 0 \implies \frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{0}{\delta M}$
Therefore,

$$\begin{aligned}\frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) &= 0 \\ \frac{g}{\delta x \delta y \delta p}\frac{\delta x \delta y}{g}\frac{D}{Dt}\delta p + \frac{g}{\delta x \delta y \delta p}\frac{\delta x \delta p}{g}\frac{D}{Dt}\delta y + \frac{g}{\delta x \delta y \delta p}\frac{\delta y \delta p}{g}\frac{D}{Dt}\delta x &= 0 \qquad \text{(Chain rule)} \\ \frac{1}{\delta x}\frac{D}{Dt}\delta x + \frac{1}{\delta y}\frac{D}{Dt}\delta y + \frac{1}{\delta p}\frac{D}{Dt}\delta p &= 0 \\ \frac{1}{\delta x}\delta\underbrace{\left(\frac{Dx}{Dt}\right)}_{=u} + \frac{1}{\delta y}\delta\underbrace{\left(\frac{Dy}{Dt}\right)}_{=v} + \frac{1}{\delta p}\delta\underbrace{\left(\frac{Dp}{Dt}\right)}_{=\omega} &= 0 \\ \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} &= 0\end{aligned}$$

Taking limits as $\delta x, \delta y, \delta p \rightarrow 0$

And $\delta x, \delta y$ are evaluated at constant pressure

$$\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p}}_{\text{No density, no time derivative!}} = 0$$

$$\frac{\partial \omega}{\partial p} = -\vec{\nabla} \cdot \vec{V}$$

Horizontal divergence: $\vec{\nabla} \cdot \vec{V} > 0 \implies \frac{\partial \omega}{\partial p} < 0$, vertical squashing.

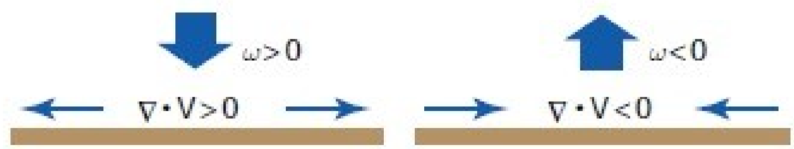


Figure 5: Algebraic signs of ω in the midtroposphere associated with convergence and divergence in the lower troposphere. [Source: Wallace, J.M. and Hobbs, P.V. (2006). Atmospheric Science: An Introductory Survey, 2nd Ed. Academic Press, pp. 483]

The thermodynamic energy equation

First law of Thermodynamics:

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \tag{2.42}$$

where $\frac{Dp}{Dt} = \omega$ and J is the diabatic heating rate; the rate of heating per unit mass due to radiation, conduction, and latent heat release.

$$\therefore c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \alpha\omega = J$$

Equation of state: $p\alpha = RT$

$$\therefore \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + \omega\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\omega = \frac{J}{c_p}$$

$$\therefore \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \omega\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T - \omega\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p}$$

Static stability parameter for the isobaric system: $S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$

$$\therefore \frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T - S_p \omega = \frac{J}{c_p}$$

It can be shown that $S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g}$

For observed lapse rate equal to the dry adiabatic lapse rate, $S_p = 0$

$$\therefore \frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T = \frac{J}{c_p}$$

If the motion is adiabatic, $J = 0$

$$\therefore \frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T = 0$$

Exercise 1: A frontal zone moves over Tshwane overnight so that the local temperature falls at a rate of $1°\text{C} \cdot \text{h}^{-1}$. The wind is blowing from the South at $10\,\text{km} \cdot \text{h}^{-1}$. The temperature is decreasing with latitude at a rate of $10°\text{C}$ per $100\,\text{km}$. Neglecting diabatic heating, and for the case of the observed lapse rate being equal to the dry adiabatic lapse rate, use the thermodynamic energy equation to describe the local rate of temperature change, and the advection of temperature over Tshwane.

Solution:

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T - S_p \omega = \frac{J}{c_p}$$

$$\therefore \frac{\partial T}{\partial t} = -\vec{V} \cdot \vec{\nabla} T$$

Left hand side: $\frac{\partial T}{\partial t} = -1°\text{C} \cdot \text{h}^{-1}$

Right hand side:

$$\begin{aligned}
-\vec{V} \cdot \vec{\nabla} T &= -(u\vec{i} + v\vec{j}) \cdot \left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} \right) \\
&= -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} \\
&= -v\frac{\partial T}{\partial y} \qquad \text{(since } u = 0\text{)} \\
&= -(10\,\text{km} \cdot \text{h}^{-1}) \left(\frac{-10°\text{C}}{100\,\text{km}} \right) \qquad (v > 0) \\
&= 1°\text{C} \cdot \text{h}^{-1}
\end{aligned}$$

In order for the left and right hand sides to be equal, the right-hand side must be reduced by $2°\text{C} \cdot \text{h}^{-1}$. Such a reduction may be caused by adiabatic cooling due to vertical advection.

$$\begin{aligned}\therefore \text{Right hand side} &= 1°\text{C} \cdot \text{h}^{-1} - 2°\text{C} \cdot \text{h}^{-1} \\ &= -1°\text{C} \cdot \text{h}^{-1} \\ &= \text{Left hand side}\end{aligned}$$

Exercise 2: Explain in words what this form of the thermodynamic energy equations represents:

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T = 0$$

Solution: It represents the balance between the local rate of temperature change and the advection of temperature.

Balanced flow

Assumptions:

1. flows are steady state (i.e. time independent)
2. no vertical component of velocity

Natural coordinates

Defined by the orthogonal set of unit vectors $\vec{t}$, $\vec{n}$ and $\vec{k}$

$\vec{t}$: parallel to the horizontal velocity at each point

$\vec{n}$: normal to the horizontal velocity; positive to the left of the flow direction

$\vec{k}$: vertically upward

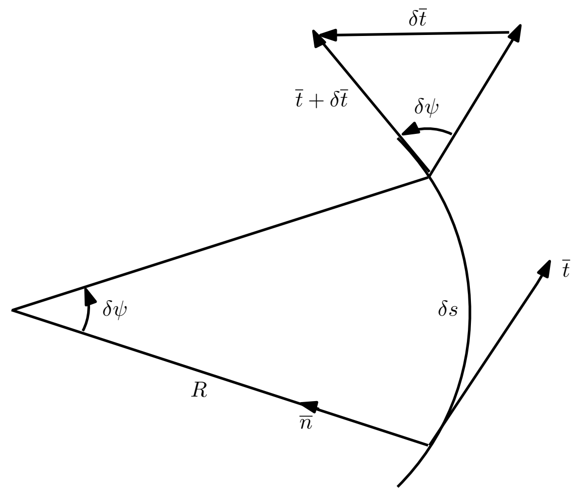
Horizontal velocity $\vec{V} = V\vec{t}$; where V is the horizontal speed, non-negative scalar

$$V \equiv \frac{Ds}{Dt}$$

where $s(x, y, t)$ is the distance along the curve of parcel.

Acceleration following the motion:

$$\begin{aligned} \frac{D\vec{V}}{Dt} &= \frac{D(V\vec{t})}{Dt} \\ &= \frac{DV}{Dt}\vec{t} + \frac{D\vec{t}}{Dt}V \end{aligned}$$



According to the figure: $\delta s = |R|\delta\psi \qquad (\text{“}s = r\theta\text{”}),$

where R is the radius of curvature following the parcel motion.

$$\delta\psi = \frac{\delta s}{|R|} = \frac{\delta\vec{t}}{|\vec{t}|} \qquad \text{(considered small triangle)}$$
$$|\vec{t}| = 1$$
$$\Longrightarrow \frac{\delta s}{|R|} = |\delta\vec{t}|$$
$$\frac{|\delta\vec{t}|}{\delta s} = \frac{1}{|R|}$$

In the limit $\delta s \to 0$, $\delta\vec{t}$ becomes parallel to $\vec{n}$

$$\Longrightarrow \frac{D\vec{t}}{Ds} = \frac{\vec{n}}{R} \qquad \text{(because } \vec{n} \text{ is a unit vector: } |\vec{n}| = 1)$$
$$\frac{D\vec{t}}{Dt} = \frac{D\vec{t}}{Ds}\frac{Ds}{Dt} = \frac{\vec{n}}{R}V \qquad \left(V \equiv \frac{Ds}{Dt}\right)$$
$$\Longrightarrow \frac{D\vec{V}}{Dt} = \vec{t}\frac{DV}{Dt} + V\left(\frac{\vec{n}}{R}V\right)$$
$$\frac{D\vec{V}}{Dt} = \vec{t}\frac{DV}{Dt} + \vec{n}\frac{V^2}{R} \tag{3.8}$$

where:

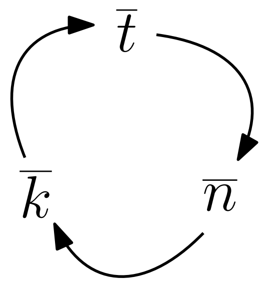
$\frac{D\vec{V}}{Dt}$ is the acceleration following the motion.

$\vec{t}\frac{DV}{Dt}$ is the rate of change of speed of the air parcel.

$\vec{n}\frac{V^2}{R}$ is the centripetal acceleration due to curvature of trajectory.

Acceleration due to **Coriolis force**:

$$
\begin{aligned}
&= -f\vec{k} \times \vec{V} \\
&= -f\vec{k} \times V\vec{t} \\
&= -fV\vec{n}
\end{aligned}
$$



Horizontal pressure gradient $= -\vec{\nabla}_p \Phi$

Change of geopotential along the curve $= \dfrac{\partial \Phi}{\partial s}$

Change of geopotential perpendicular to the curve $= \dfrac{\partial \Phi}{\partial n}$

In natural coordinate system: $= \vec{\nabla}_p \Phi = \dfrac{\partial \Phi}{\partial s}\vec{t} + \dfrac{\partial \Phi}{\partial n}\vec{n}$

Since $\dfrac{D\vec{V}}{Dt} = \underbrace{-f\vec{k} \times \vec{V} - \vec{\nabla}_p \Phi}_{(3.2)} = -fV\vec{n} - \left(\vec{t}\dfrac{\partial \Phi}{\partial s} + \vec{n}\dfrac{\partial \Phi}{\partial n}\right)$

And $\dfrac{D\vec{V}}{Dt} = \vec{t}\dfrac{DV}{Dt} + \vec{n}\dfrac{V^2}{R}$

$$\underbrace{\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}}_{(3.9)} \qquad \text{and} \qquad \frac{V^2}{R} = -fV - \frac{\partial \Phi}{\partial n} \implies \underbrace{\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}}_{(3.10)}$$

$\dfrac{DV}{Dt} = -\dfrac{\partial \Phi}{\partial s}$: force balance **parallel** to the direction of flow.

$\dfrac{V^2}{R} + fV = -\dfrac{\partial \Phi}{\partial n}$: force balance **normal** to the direction of flow.

For motion parallel to geopotential height then Φ remains unchanged:

$$
\begin{aligned}
&\frac{\partial \Phi}{\partial s} = 0 \\
\therefore\ &\frac{DV}{Dt} = 0 \\
&\implies \text{Speed is constant following the motion}
\end{aligned}
$$

If the geopotential gradient normal to the direction of motion is constant along a trajectory

$$\frac{\partial \Phi}{\partial n} = 0$$

$$\therefore \frac{V^2}{R} + fV = 0$$

$$\therefore R = \frac{-V^2}{fV} = \frac{-V}{f}$$

$\implies$ radius of curvature, R, is constant.

Geostrophic flow

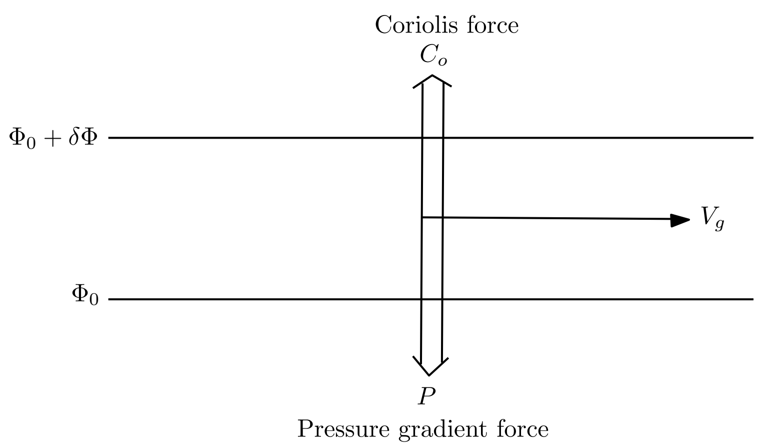
Geostrophic motion: flow in a straight line $(R \to \pm\infty)$ parallel to height contours.

For $R \to \pm\infty, \dfrac{V^2}{R} \to 0$

In geostrophic motion the horizontal components of the Coriolis force and pressure gradient force are in **exact balance**, thus $V = V_g$

$$\therefore 0 + fV = fV_g = -\frac{\partial \Phi}{\partial n} \tag{3.11}$$

The balance



The actual wind can be in exact geostrophic motion only if the height contours are parallel to latitude circles.

Although the geostrophic wind is generally a good approximation to the actual wind in extra-tropical synoptic-scale disturbances, in some special cases this is not true!

In the Southern Hemisphere, the geopotential values are smaller on the right side than on the left side relative to the direction of the geostrophic wind. The pressure gradient force is therefore directed towards the right of the wind. In order to manifest a particularly simple balance of horizontal forces on the flow, the Coriolis force associated with this wind is directed to the left.

Inertial flow

If the geopotential field is uniform on an isobaric surface so that the horizontal pressure gradient vanishes $\left(\frac{\partial \Phi}{\partial n} = 0\right)$:

$$\frac{V^2}{R} + fV = 0 \tag{3.12}$$

(3.12): Coriolis force and centrifugal force are balanced.

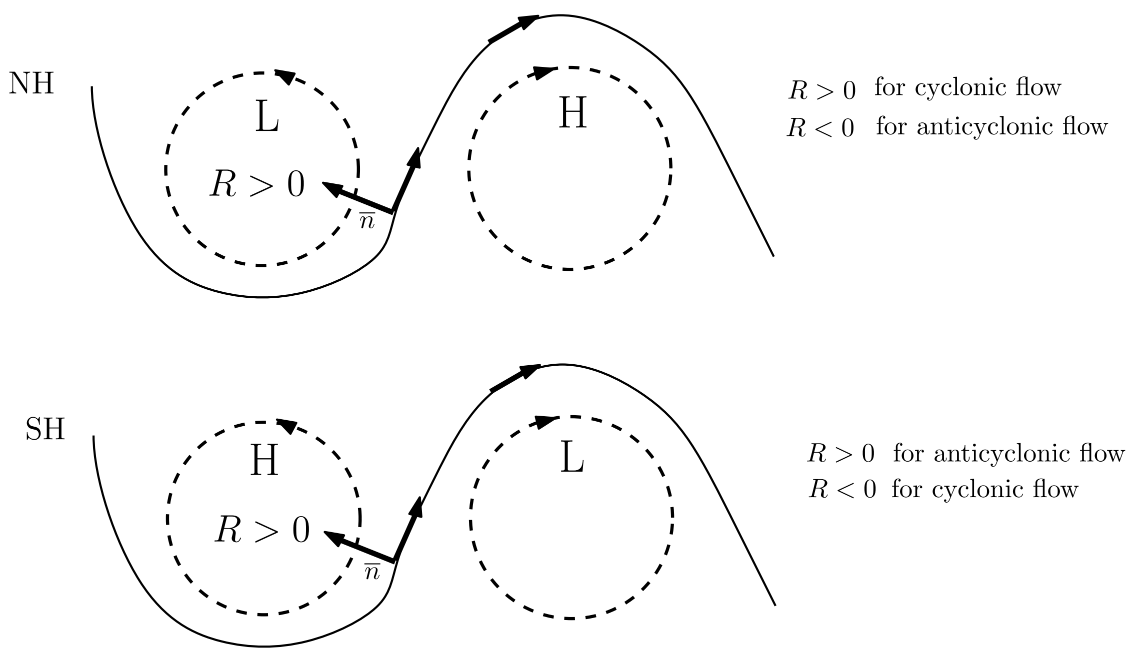
$$R = -\frac{V}{f}$$

<u>**NOTE**</u>:

1) In <u>**the atmosphere**</u> motions are nearly always generated and maintained by pressure gradient forces
2) The condition of uniform pressure required for pure inertial flow <u>**rarely exist**</u>

Balanced flow

Consider the natural coordinate system where the unit vector $\vec{n}$ is normal to the horizontal velocity and is positive to the left of the flow direction. This configuration applies to both hemispheres.



$R > 0$ when the centre of curvature is in the positive $\vec{n}$ direction.

Regarding $\dfrac{\partial\Phi}{\partial n}$:

For the NH:

For LOW pressure system

$$\begin{aligned} \delta\Phi &< 0 \qquad &&\text{(geopotential decreases towards centre)} \\ \delta n &> 0 \qquad &&(\vec{n}\text{ pointing towards centre of low}) \\ \therefore \frac{\partial\Phi}{\partial n} &< 0 \end{aligned}$$

For HIGH pressure system

$$\begin{aligned} \delta\Phi &> 0 \qquad &&\text{(geopotential increases towards centre)} \\ \delta n &< 0 \qquad &&(\vec{n}\text{ pointing towards centre of low}) \\ \therefore \frac{\partial\Phi}{\partial n} &< 0 \end{aligned}$$

For the SH:

For LOW pressure system $\delta\Phi < 0, \delta n < 0$ therefore $\dfrac{\partial\Phi}{\partial n} > 0$

For HIGH pressure system $\delta\Phi > 0, \delta n > 0$ therefore $\dfrac{\partial\Phi}{\partial n} > 0$

Cyclostrophic flow

If the horizontal scale of an atmospheric disturbance is small enough, the Coriolis force may be neglected when compared with the centrifugal force and the pressure gradient force:

Centrifugal force $\dfrac{V^2}{R} \gg fV$

Pressure gradient force $\dfrac{\partial\Phi}{\partial n} \gg fV$

From Eq. (3.10):

$$\begin{aligned} \frac{V^2}{R} &= -\frac{\partial\Phi}{\partial n} \\ V &= \left(-R\frac{\partial\Phi}{\partial n}\right)^{1/2}, \qquad &&\text{the cyclostrophic wind speed.} \end{aligned}$$

Case 1: $R > 0$ and $\dfrac{\partial\Phi}{\partial n} > 0$

$$\begin{gathered} R\frac{\partial\Phi}{\partial n} > 0 \\ \therefore -R\frac{\partial\Phi}{\partial n} < 0 \\ \Longrightarrow \text{ Negative root, } V \text{ physically impossible} \end{gathered}$$

Case 2: $R < 0$ and $\dfrac{\partial \Phi}{\partial n} > 0$

$$R\frac{\partial \Phi}{\partial n} < 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} > 0$$
$$\implies \text{Positive root, } V \text{ physically possible}$$

Case 3: $R > 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

$$R\frac{\partial \Phi}{\partial n} < 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} > 0$$
$$\implies \text{Positive root, } V \text{ physically possible}$$

Case 4: $R < 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

$$R\frac{\partial \Phi}{\partial n} > 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} < 0$$
$$\implies \text{Negative root, } V \text{ physically impossible}$$

The mathematically positive roots of the speed of the cyclostrophic wind correspond to only two physically possible solutions:

$$R < 0 \text{ and } \frac{\partial \Phi}{\partial n} > 0 \quad \text{(Case 2)}$$

and

$$R > 0 \text{ and } \frac{\partial \Phi}{\partial n} < 0 \quad \text{(Case 3)}$$

Consider the figures on the next page. Since the Coriolis force is not a factor, around lows, cyclostrophic winds can turn either clockwise or counterclockwise. As discussed in the balanced flow section, $\vec{n}$ is positive to the left of the flow direction, $R > 0$ when curvature centre is in $\vec{n}$ direction.

Therefore,

NH: Cyclonic flow $R > 0$ and anti-cyclonic flow $R < 0$

SH: Cyclonic flow $R < 0$ and anti-cyclonic flow $R > 0$

Regarding $\dfrac{\partial \Phi}{\partial n}$:

Since we are dealing here with (intense) low pressure systems $\delta\Phi < 0$ for cyclonic and anti-cyclonic flow, and for both hemispheres.

NH: Cyclonic flow $\delta n > 0$ therefore $\dfrac{\partial \Phi}{\partial n} < 0$

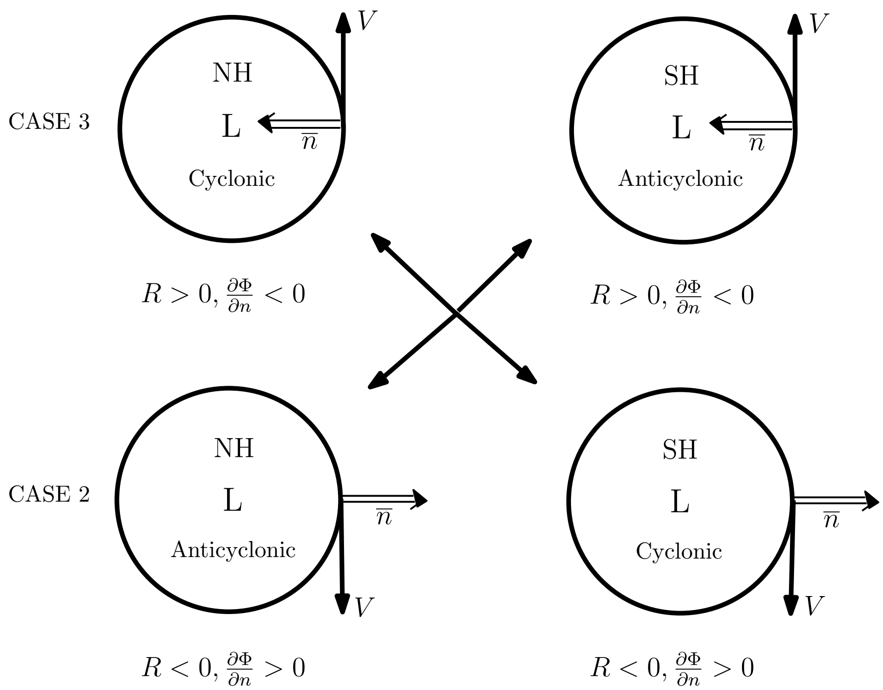
SH: Cyclonic flow $\delta n < 0$ therefore $\dfrac{\partial \Phi}{\partial n} > 0$

NH: anti-cyclonic flow $\delta n < 0$ therefore $\dfrac{\partial \Phi}{\partial n} > 0$

SH: anti-cyclonic flow $\delta n > 0$ therefore $\dfrac{\partial \Phi}{\partial n} < 0$

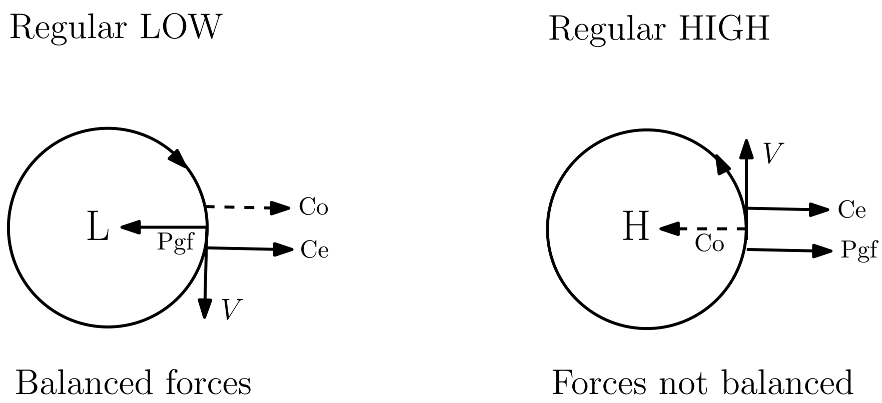
Cyclostrophic wind classification:

$\dfrac{\partial \Phi}{\partial n}$ $(+/-)$	$R > 0$	$R < 0$
Positive	Case 1: unphysical	Case 2: physical NH: anti-cyclonic; SH: cyclonic
Negative	Case 3: physical NH: cyclonic; SH: anti-cyclonic	Case 4: unphysical



Exercise: By means of drawing circular symmetric motion figures, explain why there can be no cyclostrophic balance around a small high pressure centre.

Solution:



For cyclostrophic flow, the Coriolis force becomes negligible compared with other two forces. For a LOW, there can still be balance. In this case between Ce and Pgf. However, for a HIGH, Ce and Pgf point in the same direction. So no balance is possible here.

The gradient wind approximation

Gradient Flow: Horizontal frictionless flows that is parallel to the height contours so that the tangential acceleration vanishes, i.e. $\frac{DV}{Dt} = 0$.

Gradient Flow is a 3-way balance among:

1) The Coriolis force

2) The centrifugal force

3) The horizontal pressure gradient force

A **gradient wind** is just the wind component parallel to the height contour that satisfies:

$$\underbrace{\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}}_{\text{the gradient wind equation}} \tag{3.10}$$

For a quadratic equation $ax^2 + bx + c = 0$, solving the equation for x

$$x = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

Therefore, solving for V in Eq. (3.10), $a = \frac{1}{R}$, $b = f$, $c = \frac{\partial \Phi}{\partial n}$

$$\begin{aligned} V &= \frac{-f \pm \left(f^2 - 4\frac{1}{R}\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}}}{\frac{2}{R}} \\ &= -\frac{fR}{2} \pm \frac{R}{2}\left(f^2 - \frac{4}{R}\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}} \\ &= -\frac{fR}{2} \pm \left(\left(\frac{R}{2}\right)^2 \left(f^2 - \frac{4}{R}\frac{\partial \Phi}{\partial n}\right)\right)^{\frac{1}{2}} \end{aligned}$$

For geostrophic flow

$$\begin{aligned} fV_g &= -\frac{\partial \Phi}{\partial n} \\ \therefore \frac{\partial \Phi}{\partial n} &= -fV_g \end{aligned}$$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} \tag{3.15}$$

Determining the **mathematically possible roots** of (3.15)

$$\underbrace{V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}}}_{\text{the gradient wind}}$$

By the geostrophic approximation $V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$ i.t.o the **pressure gradient**

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}} \tag{A}$$

Objective: Determine the cases for which the solution of (A) is both positive and real

The gradient wind approximation: SOUTHERN HEMISPHERE $(f < 0)$

Case 1: For $R > 0$ and $\frac{\partial \Phi}{\partial n} > 0$

$$fR < 0, \quad -\frac{fR}{2} > 0$$

$$R\frac{\partial \Phi}{\partial n} > 0, \quad -R\frac{\partial \Phi}{\partial n} < 0$$

Since V has to be real, $\dfrac{f^2R^2}{4} - R\dfrac{\partial\Phi}{\partial n} > 0$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{\frac{1}{2}}$$

For **positive** root:

$$V = \text{Positive value}\left(-\frac{fR}{2}\right) + \text{positive}(+\surd)$$

$$\therefore V > 0, \text{ and therefore physically possible.}$$

For **negative** root: First consider $\dfrac{f^2R^2}{4} > 0$

$$\therefore \frac{f^2R^2}{4} > \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} < 0\right)$$

$$\therefore \left(\frac{f^2R^2}{4}\right)^{1/2} > \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering negative roots:

$$-\left(\frac{f^2R^2}{4}\right)^{1/2} < -\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} < -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V > -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} - \underbrace{\left|\frac{fR}{2}\right|}_{\frac{fR}{2} < 0 \text{ from } *} = -\frac{fR}{2} - \left(-\frac{fR}{2}\right) = 0$$

$$\therefore V > 0, \text{ and therefore physically possible.}$$

*

$\lvert x\rvert = x$	for	$x > 0$
$\lvert x\rvert = -x$	for	$x < 0$

Case 2: For $R < 0$ and $\dfrac{\partial\Phi}{\partial n} > 0$

$$fR > 0, \;\; -\frac{fR}{2} < 0$$

$$R\frac{\partial \Phi}{\partial n} < 0, \quad -R\frac{\partial \Phi}{\partial n} > 0$$

For real V, $\dfrac{f^2R^2}{4} - R\dfrac{\partial \Phi}{\partial n} > 0$

For **negative** root:

$$\begin{aligned} V &= -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{1/2} \\ &= \text{negative value} - \text{positive value} \\ \therefore V &< 0, \text{and therefore physically impossible.} \end{aligned}$$

For **positive** root: $\dfrac{f^2R^2}{4} > 0$

$$\frac{f^2R^2}{4} < \frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial \Phi}{\partial n} > 0\right)$$

$$\therefore \left(\frac{f^2R^2}{4}\right)^{1/2} < \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{1/2}$$

But we are considering positive roots:

$$+\left(\frac{f^2R^2}{4}\right)^{1/2} < +\left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{1/2}$$

$$\begin{aligned} &\therefore -\frac{fR}{2} + \left(\frac{f^2R^2}{4}\right)^{1/2} < -\frac{fR}{2} + \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{1/2} = V \\ &\therefore V > -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} + \left|\frac{fR}{2}\right| = -\frac{fR}{2} + \frac{fR}{2} \\ &\therefore V > 0, \text{and therefore physically possible.} \end{aligned}$$

Case 3: For $R > 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

$$fR < 0, \quad -\frac{fR}{2} > 0$$

$$R\frac{\partial \Phi}{\partial n} < 0, \quad -R\frac{\partial \Phi}{\partial n} > 0$$

Since V has to be real, $\dfrac{f^2R^2}{4} - R\dfrac{\partial \Phi}{\partial n} > 0$

For **positive** root:

$$V = \text{positive value}\left(-\frac{fR}{2}\right) + \text{positive value}(+\surd)$$

$$\therefore V > 0, \text{and therefore physically possible.}$$

For **negative** root: $\dfrac{f^2R^2}{4} > 0$

$$\therefore \frac{f^2R^2}{4} < \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} > 0\right)$$

$$\therefore \left(\frac{f^2R^2}{4}\right)^{1/2} < \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering negative roots:

$$-\left(\frac{f^2R^2}{4}\right)^{1/2} > -\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} > -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V < -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} - \left|\frac{fR}{2}\right| = -\frac{fR}{2} - \left(-\frac{fR}{2}\right) \qquad \left(\text{since } \frac{fR}{2} < 0\right)$$

$\therefore V < 0$, and therefore physically impossible.

Case 4: For $R < 0$ and $\dfrac{\partial\Phi}{\partial n} < 0$

$$fR > 0, \quad -\frac{fR}{2} < 0$$

$$R\frac{\partial\Phi}{\partial n} > 0, \quad -R\frac{\partial\Phi}{\partial n} < 0$$

For real V, $\dfrac{f^2R^2}{4} - R\dfrac{\partial\Phi}{\partial n} > 0$

For **negative** root:

$$V = -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$= \text{negative value} - \text{positive value}$$

$$\therefore V < 0, \text{ and therefore physically impossible.}$$

For **positive** root: $\dfrac{f^2R^2}{4} > 0$

$$\frac{f^2R^2}{4} > \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} < 0\right)$$

$$\therefore \left(\frac{f^2R^2}{4}\right)^{1/2} > \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering positive roots:

$$+\left(\frac{f^2R^2}{4}\right)^{1/2} > +\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} + \left(\frac{f^2R^2}{4}\right)^{1/2} > -\frac{fR}{2} + \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V < -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} + \left|\frac{fR}{2}\right| = -\frac{fR}{2} + \frac{fR}{2}$$

$\therefore V < 0$, and therefore physically impossible.

The following table is a summary of the four cases

Gradient wind classification in the Southern Hemisphere

$\frac{\partial\Phi}{\partial n}$ $(+/-)$	$R > 0$ (anti-cyclonic)	$R < 0$ (cyclonic)
Positive	Case 1 $+$ root : physical $-$ root : physical	Case 2 $+$ root : physical $-$ root : unphysical
Negative	Case 3 $+$ root : physical $-$ root : unphysical	Case 4 $+$ root : unphysical $-$ root : unphysical

For **cyclonic** flow ($R < 0$) the only physically possible configuration is:

$R < 0$ and $\frac{\partial\Phi}{\partial n} > 0$ (similar to result in the cyclostrophic wind classification section)

$\frac{\partial\Phi}{\partial n} > 0$ makes sense since $\delta\Phi < 0$ and $\delta n < 0$. Also consider the geostrophic wind equation

$$V_g = -\frac{1}{f}\frac{\partial\Phi}{\partial n} \tag{3.11}$$

$$\therefore V_g = -\frac{1}{\text{(neg)}}\text{(pos)} > 0$$

For **anti-cyclonic** flow ($R > 0$), $\frac{\partial\Phi}{\partial n}$ can be either positive or negative.

So, for $\frac{\partial\Phi}{\partial n} > 0$: $V_g = -\frac{1}{\text{(neg)}}\text{(pos)} > 0$ as was found for cyclonic flow.

Case 3: However, for $\frac{\partial\Phi}{\partial n} < 0$: $V_g = -\frac{1}{\text{(neg)}}\text{(neg)} < 0$, which is anti-geostrophic.

Again consider:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \tag{3.10}$$

The geostrophic wind is defined by

$$fV_g = -\frac{\partial \Phi}{\partial n} \tag{3.11}$$

$$\begin{aligned}
\therefore \frac{V^2}{R} + fV &= fV_g \\
\frac{1}{fV}\left(\frac{V^2}{R} + fV\right) &= \frac{1}{fV}(fV_g) \\
\frac{V}{fR} + 1 &= \frac{V_g}{V}, \text{the ratio of the geostrophic wind to the gradient wind.}
\end{aligned}$$

From the balanced flow section:

Northern Hemisphere ($f > 0$):

$$\begin{aligned}
R > 0 &\ \text{ for cyclonic flow} \\
R < 0 &\ \text{ for anti-cyclonic flow} \\
\therefore Rf > 0 &\ \text{ for cyclonic flow} \\
Rf < 0 &\ \text{ for anti-cyclonic flow}
\end{aligned}$$

Southern Hemisphere ($f < 0$):

$$\begin{aligned}
R < 0 &\ \text{ for cyclonic flow} \\
R > 0 &\ \text{ for anti-cyclonic flow} \\
\therefore Rf > 0 &\ \text{ for cyclonic flow} \\
Rf < 0 &\ \text{ for anti-cyclonic flow}
\end{aligned}$$

Typical values for V,f and R: $5\,\mathrm{m \cdot s^{-1}}$, $10^{-4}\,\mathrm{s^{-1}}$ and $500\,\mathrm{km}$.

$$\therefore \frac{V}{fR} = \frac{5\,\mathrm{m \cdot s^{-1}}}{10^{-4}\,\mathrm{s^{-1}}\,500000\,\mathrm{m}} = 0.1$$

For cyclonic flow (both hemispheres):

$$\begin{aligned}
\frac{V_g}{V} &= 1 + 0.1 = 1.1 \\
\therefore V_g &= 1.1 \times V \\
\implies V_g &> V, \text{ sub-geostrophic}
\end{aligned}$$

For cyclonic flow, the geostrophic wind is stronger than the gradient wind.

For anti-cyclonic flow (both hemispheres):

$$\frac{V_g}{V} = 1 - 0.1 = 0.9$$
$$\therefore V_g = 0.9 \times V$$
$$\implies V_g < V, \text{ super-geostrophic}$$

For anti-cyclonic flow, the gradient wind is stronger than the geostrophic wind.

Therefore, the geostrophic wind is an overestimate of the balanced wind in a region of cyclonic curvature, and an underestimate in a region of anti-cyclonic curvature.

Next we want to illustrate the force balances for the permitted solutions. For this purpose we want to determine

1) the direction of these forces
2) their relative sizes

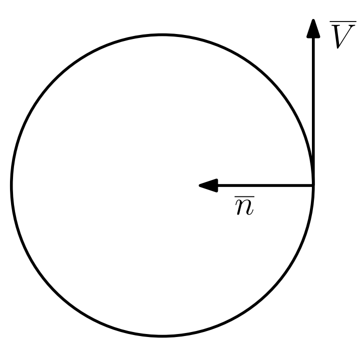
Take note of the following:

1) the centrifugal force (C_e) always points outwards,
2) the pressure gradient force (Pgf) is always from a high to a low pressure system,
3) the Coriolis force (C_o) is to the left of the motion in the Southern Hemisphere, and
4) the vector $\vec{n}$ is positive to the left and perpendicular to $\vec{V}$.

First, determine the sign of R: $R > 0 \implies$ anti-cyclone, or $R < 0 \implies$ cyclone.

Second, draw a circular flow structure with $\vec{n}$ perpendicular and to the left of $\vec{V}$.

Example, for $R > 0$ (anti-cyclone):

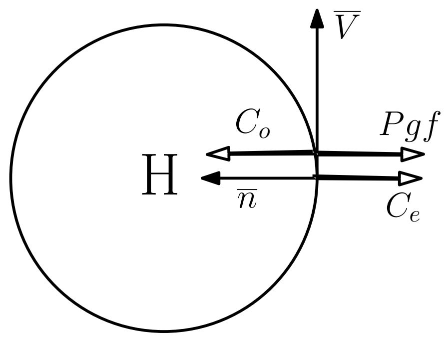


Third, consider the sign of $\dfrac{\partial \Phi}{\partial n}$

For example, for $\dfrac{\partial \Phi}{\partial n} > 0$:

For the circular flow above, $\delta n > 0$ in the direction towards the centre of the circle. Therefore $\delta \Phi > 0$ in the direction of the centre of the circle, which means that here we are dealing with a high pressure system.

Now we can complete the circular flow structure above by including the appropriate forces:



This flow represents Case 1 $\left(R > 0, \dfrac{\partial \Phi}{\partial n} > 0\right)$

Take note: For now we are not concerned with the relative sizes of the vectors that represent the three forces, but only with the direction of these forces. The discussion of the relative sizes is still to follow.

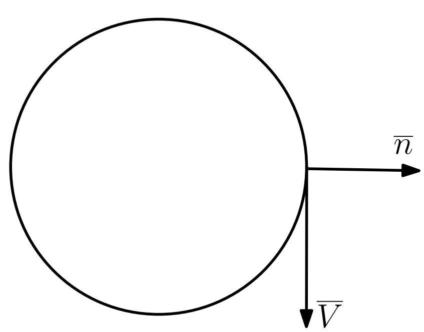
C_o : Coriolis force

C_e : Centrifugal force

Pgf : Pressure gradient force

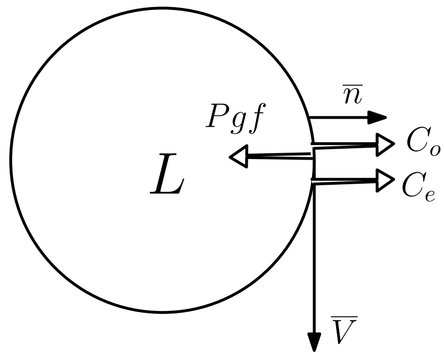
For Case 2, $R < 0$ and $\dfrac{\partial \Phi}{\partial n} > 0$

Owing to the sign of R (< 0), the circular flow will be cyclonic:



In the direction towards centre of circle, $\delta n < 0$ and $\dfrac{\delta \Phi}{\delta n} > 0$, therefore $\delta \Phi < 0$, which means we are dealing with a low pressure system.

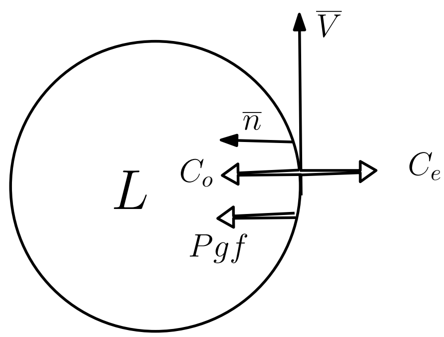
Now complete the circular flow structure by including the appropriate forces:



For Case 3, $R > 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

Since $R > 0$, the circular flow is similar to Case 1 (anti-cyclonic) and $\delta n > 0$.

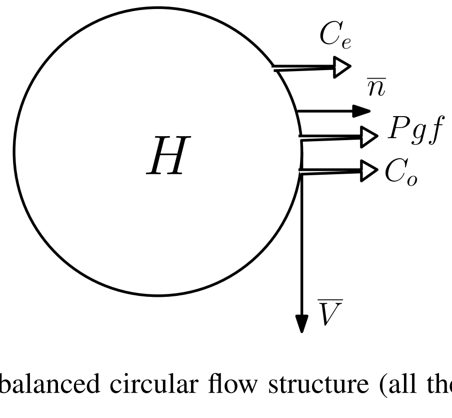
However $\dfrac{\delta \Phi}{\delta n} < 0$, therefore $\delta \Phi < 0$, which means we are dealing with a low pressure system that rotates anti-cyclonically.



For Case 4, $R < 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

Since $R < 0$, the circular flow is similar to Case 2 (cyclonic) and $\delta n < 0$.

However $\dfrac{\delta \Phi}{\delta n} < 0$, therefore $\delta \Phi > 0$, which means we are dealing with a high pressure system that rotates cyclonically.



This configuration leads to an unbalanced circular flow structure (all the forces are in the same direction) that is not physically possible.

We have now managed to determine the direction of the three forces for various circular flow structures. Next, we will attempt to obtain insight into the relative sizes of the forces associated with the flow structures.

Consider the following parameter values representative of extra-tropical circulation in the Southern Hemisphere:

$$\begin{aligned} f &= -10^{-4}\,\mathrm{s}^{-1} \\ |R| &= 10^{6}\,\mathrm{m} \\ \left|\frac{\partial\Phi}{\partial n}\right| &= 10^{-3}\,\mathrm{m\,s}^{-2} \end{aligned}$$

Calculate the gradient wind speeds by using (3.15) of Holton 4 for each of the four cases.

Case 1: $R > 0$ and $\dfrac{\delta\Phi}{\delta n} > 0$

Remember that for this case, V has two physical solutions, one associated with the a positive root of (3.15) and one with a negative root.

The first term on the right of (3.15) is $-\dfrac{fR}{2}$. Since for this case $-\dfrac{fR}{2} > 0$,

$$\begin{aligned} V &> -\frac{fR}{2} \text{ for a positive root} \\ V &< -\frac{fR}{2} \text{ for a negative root, as shown with the calculation of } V \text{ below.} \end{aligned}$$

By using the parameters above

$$\begin{aligned} -\frac{fR}{2} &= \frac{-\left(-10^{-4}\,\mathrm{s}^{-1}\right)\left(+10^{6}\,\mathrm{m}\right)}{2} \\ &= 50\,\mathrm{m\,s}^{-1} \end{aligned}$$

$$\begin{aligned} \therefore V &= 50\,\mathrm{m\,s}^{-1} \pm \left(\frac{\left(-10^{-4}\,\mathrm{s}^{-1}\right)^{2}\left(+10^{6}\,\mathrm{m}\right)^{2}}{4} - \left(+10^{6}\,\mathrm{m}\right)\left(10^{-3}\,\mathrm{m\,s}^{-2}\right)\right)^{\frac{1}{2}} \\ &= 50\,\mathrm{m\,s}^{-1} \pm 38.73\,\mathrm{m\,s}^{-1} \end{aligned}$$

For a positive root, $V = 50 + 38.73 = 88.73\,\mathrm{m\,s}^{-1} > 50$, an anomalous high and confirms that $V > -\dfrac{fR}{2}$

This very large speed associated with a positive root makes the circular anti-cyclonic flow anomalous. The high speed of the gradient wind will result in the centrifugal and Coriolis force becoming large.

To demonstrate this point, consider (3.10): $\dfrac{V^2}{R} + fV = -\dfrac{\partial\Phi}{\partial n}$

$$\frac{V^2}{R} = \text{centripetal acceleration or centrifugal force per unit mass } (C_e)$$
$$fV = \text{Coriolis force per unit mass } (C_o)$$
$$\frac{\partial \Phi}{\partial n} = \text{horizontal pressure gradient force (per unit mass; } Pgf)$$

Take note that centripetal force and centrifugal force are the exact same force, just in opposite directions because they are experienced from different forms of reference (i.e, inward vs. outward).

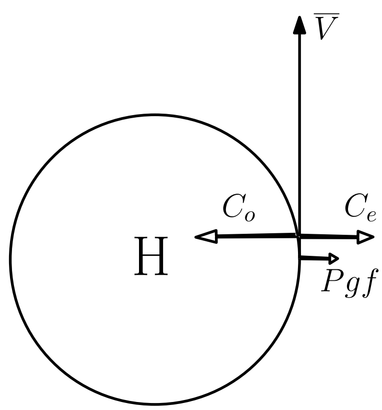
Next we will calculate the absolute strength of these forces by using the parameter values presented above.

For Case 1's positive root:

$$C_e = \frac{\left(88.73\,\mathrm{m\,s^{-1}}\right)^2}{10^6\,\mathrm{m}} = 7.873 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}}$$
$$C_o = \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(88.73\,\mathrm{m\,s^{-1}}\right) = 8.873 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}}$$
$$\frac{\partial \Phi}{\partial n} = 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}$$

Therefore, by using representative numbers we demonstrated that for Case 1 and positive roots, the centrifugal and Coriolis forces are similar in strength, and about one order of magnitude stronger than the pressure gradient force.

The circular flow structure for an anomalous high may take the following form:

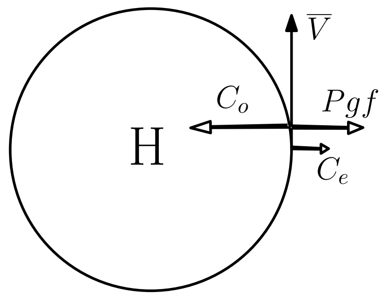


For Case 1's negative root, $V = 50 - 38.73 = 11.27\,\mathrm{m\,s^{-1}}$, a regular high and confirms that $V < -\frac{fR}{2}$

$$C_e = \frac{\left(11.27\,\mathrm{m\,s^{-1}}\right)^2}{10^6\,\mathrm{m}} = 1.270 \times 10^{-4}\,\mathrm{m\,s^{-2}} \sim 10^{-4}\,\mathrm{m\,s^{-2}}$$
$$C_o = \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(11.27\,\mathrm{m\,s^{-1}}\right) = 1.127 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-3}\,\mathrm{m\,s^{-2}}$$
$$Pgf = 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}$$

The much more realistic gradient wind speed associated with a negative root does not make the centrifugal force large as was the case with the positive root, but results in the Coriolis and pressure gradient forces becoming of similar magnitude.

The circular flow structure for such a regular high may take the following form:



For Case 2, only a positive root is associated with a physical solution for the gradient wind.

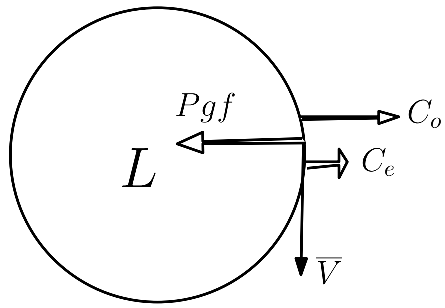
Remember that for this case $R < 0$ (cyclonic) and $\dfrac{\partial \Phi}{\partial n} > 0$

Therefore, $-\dfrac{fR}{2} < 0$ and is equal to $-50\,\mathrm{m\,s^{-1}}$

$$
\begin{aligned}
\therefore V &= -50\,\mathrm{m\,s^{-1}} + \left(\frac{\left(-10^{-4}\,\mathrm{s^{-1}}\right)^2 \left(-10^{6}\,\mathrm{m}\right)^2}{4} - \left(-10^{6}\,\mathrm{m}\right)\left(10^{-3}\,\mathrm{m\,s^{-2}}\right) \right)^{\frac{1}{2}} \\
&= -50\,\mathrm{m\,s^{-1}} + 59.16\,\mathrm{m\,s^{-1}} \\
&= 9.16\,\mathrm{m\,s^{-1}}, \text{ a regular low} \\
\therefore V &> -\frac{fR}{2}
\end{aligned}
$$

$$
\begin{aligned}
C_e &= \frac{\left(9.16\,\mathrm{m\,s^{-1}}\right)^2}{10^{6}\,\mathrm{m}} = 8.390 \times 10^{-5}\,\mathrm{m\,s^{-2}} \sim 10^{-4}\,\mathrm{m\,s^{-2}} \\
C_o &= \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(9.16\,\mathrm{m\,s^{-1}}\right) = 9.16 \times 10^{-4}\,\mathrm{m\,s^{-2}} \sim 10^{-3}\,\mathrm{m\,s^{-2}} \\
Pgf &= 10^{-3}\,\mathrm{m\,s^{-2}}, \text{ given}
\end{aligned}
$$

The circular flow structure for a regular low may take the following form:



For Case 3, only a positive root is associated with a physical solution for the gradient wind.

For this case, $R > 0$ (anti-cyclonic) and $\dfrac{\partial \Phi}{\partial n} < 0$

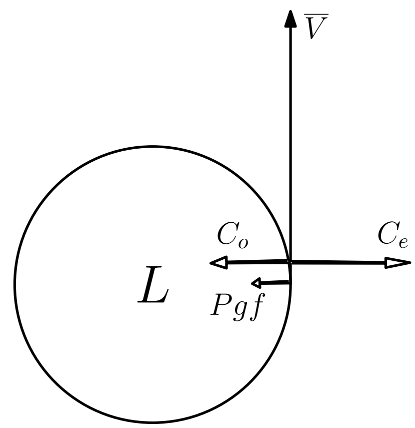
Therefore, $-\dfrac{fR}{2} > 0$ and is equal to $50\,\mathrm{m\,s^{-1}}$

$$
\begin{aligned}
\therefore V &= 50\,\mathrm{m\,s^{-1}} + \left(\frac{\left(-10^{-4}\,\mathrm{s^{-1}}\right)^2\left(+10^{6}\,\mathrm{m}\right)^2}{4} - \left(+10^{6}\,\mathrm{m}\right)\left(-10^{-3}\,\mathrm{m\,s^{-2}}\right)\right)^{\frac{1}{2}} \\
&= 50\,\mathrm{m\,s^{-1}} + 59.16\,\mathrm{m\,s^{-1}} \\
&= 109.16\,\mathrm{m\,s^{-1}} \\
\therefore V &> -\frac{fR}{2}
\end{aligned}
$$

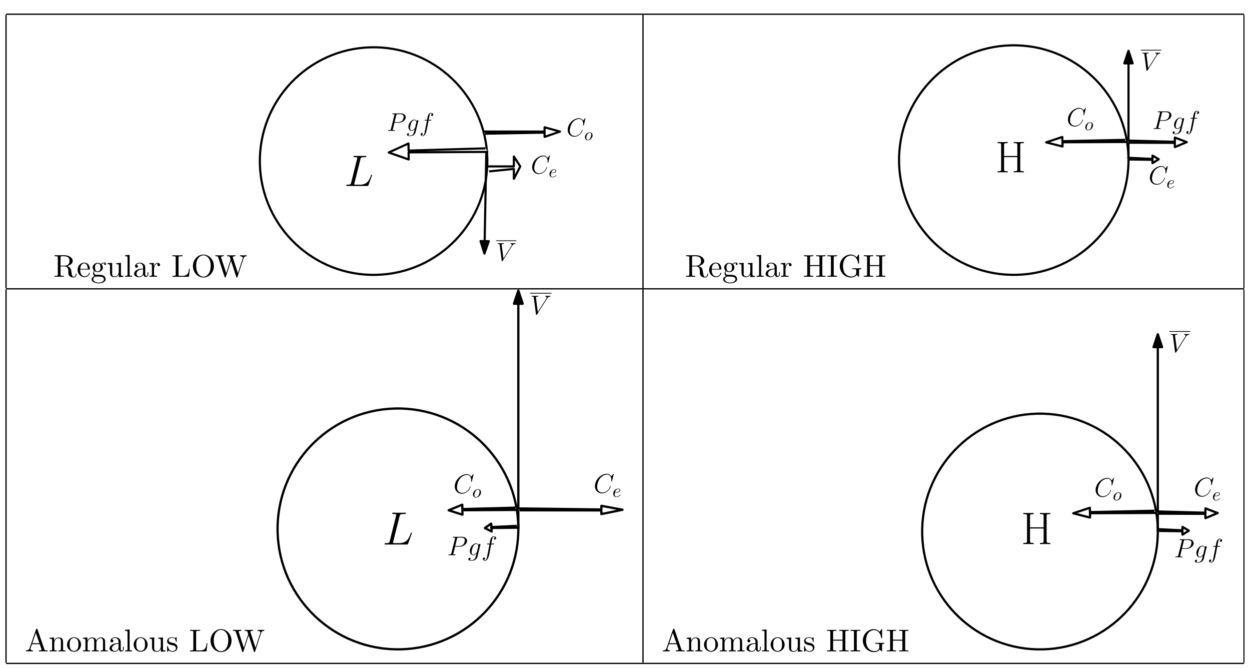
Take note that for this case we are dealing with a low pressure system that rotates counter-clockwise (anti-cyclonically) in the Southern Hemisphere! This flow orientation and high gradient wind speed makes this circular flow an anomalous low.

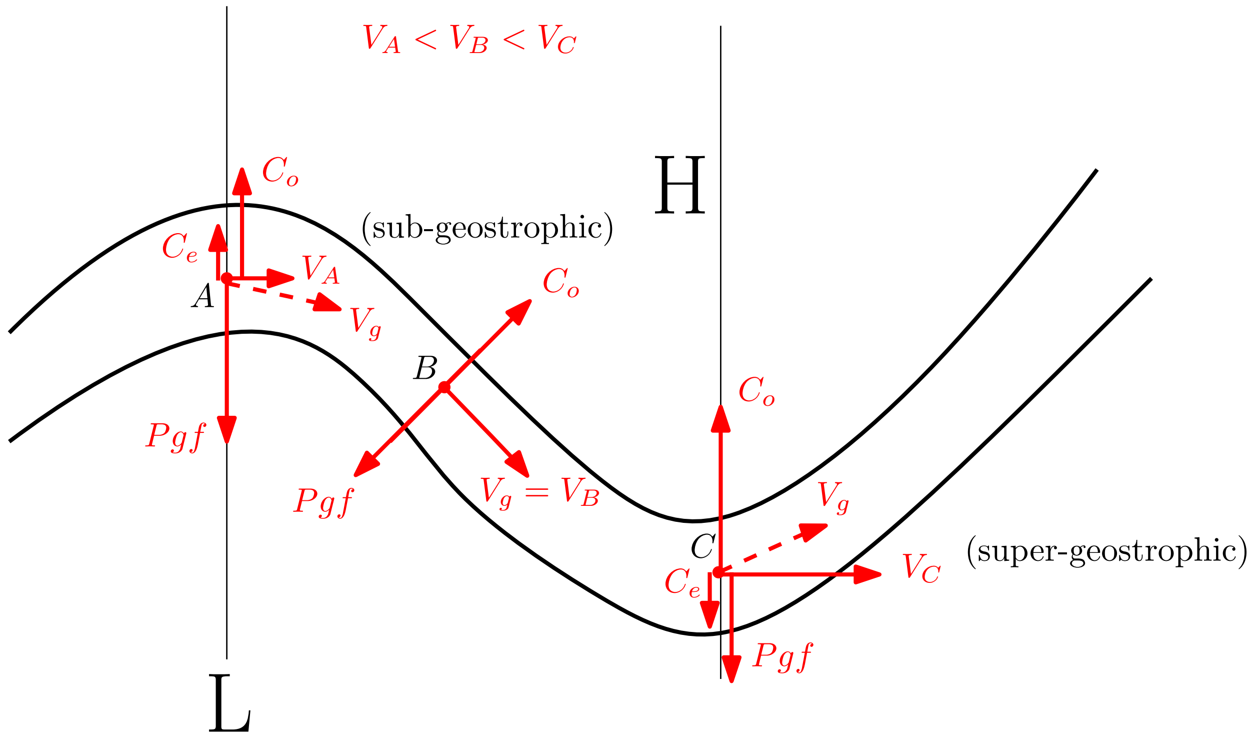
$$
\begin{aligned}
C_e &= \frac{\left(109.16\,\mathrm{m\,s^{-1}}\right)^2}{10^{6}\,\mathrm{m}} = 1.192\times10^{-2}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}} \\
C_o &= \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(109.16\,\mathrm{m\,s^{-1}}\right) = 1.092\times10^{-2}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}} \\
Pgf &= 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}
\end{aligned}
$$

The circular flow structure for an anomalous low may take the following form:


$\overline{V}$
C_o
C_e
L
Pgf

SUMMARY:


Pgf
C_o
L
C_e
$\overline{V}$
Regular LOW
$\overline{V}$
C_o
Pgf
H
C_e
Regular HIGH
$\overline{V}$
C_o
C_e
L
Pgf
Anomalous LOW
$\overline{V}$
C_o
C_e
H
Pgf
Anomalous HIGH



Consider a highly idealized trough-ridge system in the Southern Hemisphere at an isobaric level, say the level at 500 hPa. We have shown above that for a regular low pressure system the $Pgf > C_o > C_e$, and for a regular high pressure system $C_o > Pgf > C_e$. Moreover, we have already shown that for cyclonic flow $V_g > V$, and for anti-cyclonic flow $V_g < V$. The gradient wind speeds at the locations marked A (on trough line), B (in between trough and ridge lines) and C (ridge line) have gradient wind speeds of respectively V_A, V_B and V_C. Since the gradient wind speed is less than the geostrophic wind speed at the trough and is greater than the geostrophic wind speed at the ridge, we have $V_A < V_B < V_C$. The gradient wind speed therefore increases from the trough line towards the ridge line. However, in the absence of curvature (typically at point B on the idealised flow structure) C_e is absent and therefore perfect geostrophic balance is achieved before the gradient wind accelerates further towards the ridge.

Next we will try to obtain insight into the behaviour of the gradient wind speed for regular high (Case 1, negative root) and regular low (Case 2, positive root) pressure systems when only the pressure gradient force is allowed to increase incrementally (1% increase over 250 iterations), while the other two forces are kept constant. Such an increase in the pressure gradient is reminiscent of a strengthening high pressure system and a deepening low pressure system, respectively. Figure 6 shows the respective gradient wind speeds obtained by using the parameter values presented above. The gradient wind speed of the regular high pressure system becomes imaginary after 93 iterations. This result suggests that for a regular high pressure system, there may be a limit to the intensity of such a developing system. Moreover, gradient wind speeds increase indefinitely for the case of a regular low. This result, on the other hand, suggests that there may not be a limit to the depth of a low pressure centre.

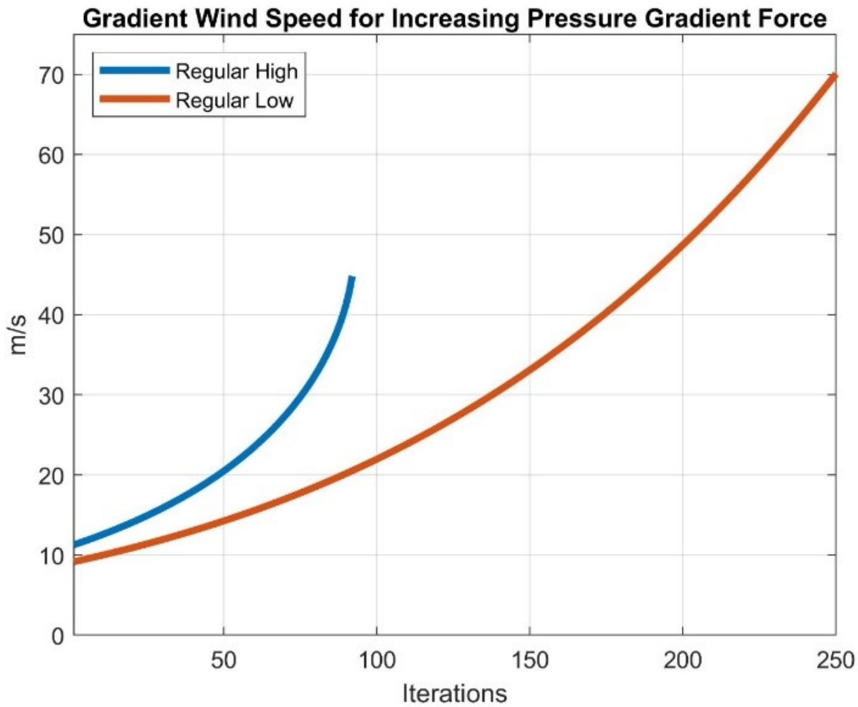


Figure 6: Gradient wind speed of regular high and regular low pressure systems by incrementally increasing the pressure gradient force by 1% for 250 iterations.

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}} \tag{A}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \tag{B}$$

An increase in the Pgf on the RHS of (B) implies that the Coriolis force and Centrifugal force on the LHS of (B) must also increase for the forces to be in balance. For a constant f and R, it follows that V must increase if the Pgf were to increase. A regular high is associated with $R > 0$, $Pgf > 0$ and a negative root in (A). Since the term $-R\frac{\partial \Phi}{\partial n} < 0$ and the term $\frac{f^2R^2}{4} > 0$, it follows that if the Pgf increases substantially whereby $\left|\frac{f^2R^2}{4}\right| < \left|-R\frac{\partial \Phi}{\partial n}\right|$, the term under the root will be negative resulting in V becoming imaginary or undefined. This is depicted in Figure 6.

Figure 7 shows the results when the pressure gradient and Coriolis forces are kept constant, but the radius of curvature, R, is allowed to respectively increase and decrease incrementally (1% change over 250 iterations). The figure shows that there is a limit to how small a high pressure system can become, since the gradient wind becomes imaginary after only 92 iterations of decreasing R. When R is allowed to increase, the gradient wind speed decreases. This result indicates that high pressure systems are generally large and have light winds. High pressure systems with small R and the resulting large gradient wind speeds are unstable and seldom occur in nature. For a low pressure system, Figure 7 shows that the effect of a varying R is likely quite small in increasing the gradient wind speed of this type of weather system. Since $R < 0$ in a regular low, it follows from (B) that the centrifugal force actually counteracts the work of the Coriolis force to balance the Pgf. An intense weather system such as a tropical cyclone therefore relies primarily on the Coriolis force and Pgf to initiate its development. However, once the tropical cyclone has reached maturity, the Pgf and centrifugal forces dominate. The following example illustrates this notion.

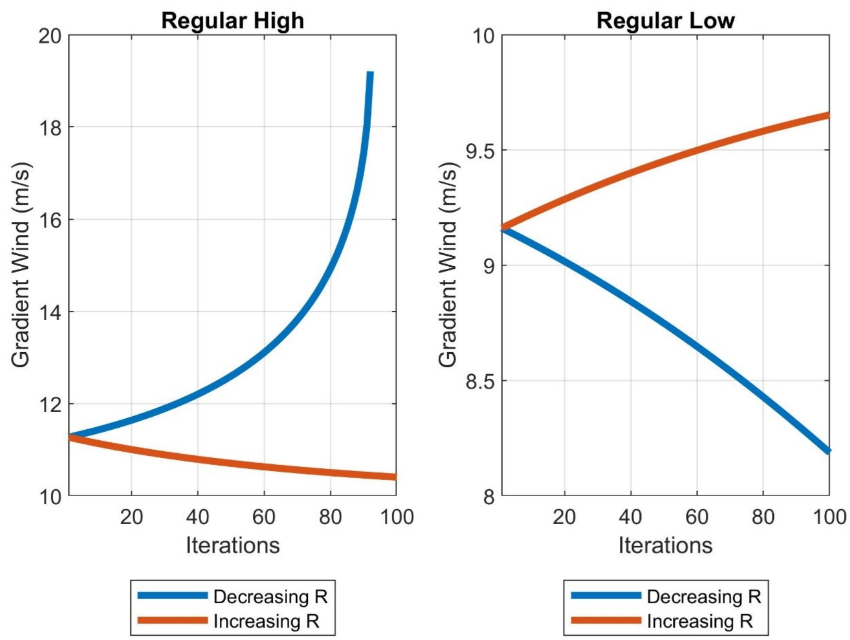


Figure 7: Gradient wind speeds of regular high and regular low pressure systems by incrementally changing the radius of curvature, R.

Consider the case of a strong tropical cyclone in the Southern Hemisphere with a central pressure of $950\,\mathrm{hPa}$ and a maximum wind speed of $60\,\mathrm{m\,s^{-1}}$. Assume that the normal pressure outside the tropical cyclone is $1010\,\mathrm{hPa}$. The centre of the storm is at $15°$S. To calculate the Pgf of this cyclone requires evaluation of the $\dfrac{\partial \Phi}{\partial n}$ term in the gradient wind equation. Since

$$\frac{\delta p}{\delta z} = -\rho g; \quad \delta\Phi = g\delta z$$
$$\therefore \frac{\delta p}{\delta \Phi} g = -\rho g$$
$$\therefore \delta\Phi = -\frac{1}{\rho}\delta p$$

$$\begin{aligned}\delta\Phi = \frac{1}{\rho}\delta p = \frac{(1010-950)\times 10^2\,\mathrm{Pa}}{1.2\,\mathrm{kg\,m^{-3}}}\\ &= 5000\,\mathrm{Pa\,kg^{-1}\,m^3}\\ &= 5000\,\mathrm{N\,m^{-2}\,kg^{-1}\,m^3}\\ &= 5000\,\mathrm{kg\,m\,s^{-2}\,m^{-2}\,kg^{-1}\,m^3}\\ &= 5000\,\mathrm{m^2\,s^{-2}}\end{aligned}$$

A tropical cyclone can be considered an intermediate size system since its typical length scale of $500\,\mathrm{km}$ is larger than that of a coastal low, but smaller than that of a strong mid-latitude cyclone. Therefore, the length scale of a tropical cyclone is the vortex radius. Therefore $\frac{\delta\Phi}{\delta n} \sim \frac{\delta p}{\rho R}$, where R is the radius of curvature.

$\therefore \frac{\delta\Phi}{\delta n} \sim \frac{5000\,\mathrm{m^2\,s^{-2}}}{500\,000\,\mathrm{m}} = 0.01\,\mathrm{m\,s^{-2}}$, which is an order of magnitude stronger then the value used before.

Since a tropical cyclone always rotates clockwise in the Southern Hemisphere it cannot be an anomalous low as one of the physically possible solutions of the gradient wind equation. Therefore, tropical cyclones are regular low pressure systems with $R < 0$ and $\frac{\delta\Phi}{\delta n} > 0$. In order to estimate this cyclone's radius and the importance of the centrifugal force relative to the Coriolis force, consider the gradient wind equation:

$$\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n}, \qquad R < 0, \frac{\partial\Phi}{\partial n} > 0$$

$$\begin{aligned}f &= 2\Omega\sin(-15^\circ)\\ &= 2\cdot\left(7.292\times 10^{-5}\,\mathrm{s^{-1}}\right)\cdot\sin(-15^\circ)\\ &= -3.775\times 10^{-5}\,\mathrm{s^{-1}}\end{aligned}$$

$$\begin{aligned}\frac{V^2}{R} &= -\left(\frac{\partial\Phi}{\partial n} + fV\right)\\ \therefore R &= -V^2\left(\frac{\partial\Phi}{\partial n} + fV\right)^{-1}\\ &= -60^2\left(10^{-2} + \left(-3.775\times 10^{-5}\right)(60)\right)^{-1}\\ &= -4.654\times 10^5\,\mathrm{m}\\ &= -465.4\,\mathrm{km},\ \text{a realistic length scale!}\end{aligned}$$

$$\text{Centrifugal force:}\quad \frac{V^2}{R} = \frac{60^2(\mathrm{m\,s^{-1}})^2}{-4.654\times 10^5\,\mathrm{m}} = -7.735\times 10^{-3}\,\mathrm{m\,s^{-2}}$$

$$\text{Coriolis force:}\quad fV = -3.775\times 10^{-5}(60) = -2.265\times 10^{-3}\,\mathrm{m\,s^{-2}}$$

Therefore, $\frac{C_e}{C_o} = 3.4$. This result is in contrast to the previous calculation of the relative sizes of the three forces where, for a regular low, the Coriolis force is larger than the centrifugal force. Notwithstanding, for a tropical cyclone presented here, the centrifugal force dominates the Coriolis force by a factor of 3. However, such a cyclone is still not cyclostrophic for which the centrifugal force is expected to be an order of magnitude larger than the Coriolis force.

Exercise 1: Consider an anti-cyclone and the case of positive pressure gradient forces. At a radius of $100\,\mathrm{km}$ and associated geostrophic wind speed of $2.4\,\mathrm{m\,s^{-1}}$, calculate the gradient wind speeds. Is the ratio between the given geostrophic wind and the calculated gradient winds in agreement with super-geostrophic flow, i.e., $V_g < V$? Next, redo the gradient wind calculation, but this time double the geostrophic wind speed and interpret this result. The Coriolis parameter is $-10^{-4}\,\mathrm{s^{-1}}$.

Solution: For anti-cyclone: $R > 0$ and (given) $\frac{\partial \Phi}{\partial n} > 0$

$$\begin{aligned} R &= 100\,000\,\mathrm{m} \\ V_g &= 2.4\,\mathrm{m\,s^{-1}} \\ f &= -10^{-4}\,\mathrm{s^{-1}} \end{aligned}$$

$$\begin{aligned} V &= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} \\ &= -\frac{(-10^{-4})(100\,000)}{2} \pm \left(\frac{(-10^{-4})^2(100\,000)^2}{4} + (-10^{-4})(100\,000)(2.4)\right)^{\frac{1}{2}} \\ &= 5 \pm (25 - 24)^{\frac{1}{2}} \\ &= 5 \pm 1\,\mathrm{m\,s^{-1}} \end{aligned}$$

For positive root: $V = 6\,\mathrm{m\,s^{-1}}$

For negative root: $V = 4\,\mathrm{m\,s^{-1}}$

For anti-cyclonic flow, it has been demonstrated that:

$$V_g < V$$

Since both V solutions are greater than $2.4\,\mathrm{m\,s^{-1}}$, the given geostrophic wind speed. Therefore, the ratio between the geostrophic wind and the gradient wind is in agreement with the result.

For double geostrophic wind, $V_g = 4.8\,\mathrm{m\,s^{-1}}$

$$\begin{aligned} V &= 5 \pm (25 - 48)^{\frac{1}{2}} \\ &= 5 \pm (\text{negative value})^{\frac{1}{2}} \end{aligned}$$

Therefore, $4.8\,\mathrm{m\,s^{-1}}$ as a geostrophic wind is unrealistically high since this leads to an unphysical solution for V.

Exercise 2: When the two terms under the square root of the solved quadratic equation (3.10) are perfectly balanced (their sum equals zero), determine the ratio of the anti-cyclonic gradient wind speed to the geostrophic wind speed for the same pressure gradient.

Solution:

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}}$$

$$\begin{aligned}
\text{Given : } \quad \frac{f^2R^2}{4} + fRV_g = 0 &\implies V = -\frac{fR}{2} \\
\therefore \frac{fR}{4} + V_g &= 0 \\
\therefore -\frac{2V}{4} + V_g &= 0 \\
\therefore V &= 2V_g \\
\therefore \frac{V}{V_g} &= 2
\end{aligned}$$

Exercise 3: Show that as the pressure gradient approaches zero the gradient wind reduces to the geostrophic wind for a normal anti-cyclone [Hint: make use of this approximation: when variable x approaches zero, the square root of $1 + x$ is equal to $1 + x/2$].

Solution: Since the pressure gradient approaches zero, so does V_g because $V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$.

The following approximation has been given: when $x \to 0$, $(1 + x)^{1/2} = 1 + \frac{x}{2}$

We therefore consider the square root term of the gradient wind equation:

$$\begin{aligned}
\pm\left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} &= \pm\left[\frac{f^2R^2}{4}\left(1 + \frac{4V_g}{fR}\right)\right]^{\frac{1}{2}} \\
&= \pm\frac{fR}{2}\left(1 + \frac{4V_g}{fR}\right)^{\frac{1}{2}} \\
&= \pm\frac{fR}{2}\left(1 + \frac{1}{2}\cdot\frac{4V_g}{fR}\right)
\end{aligned}$$

$$\therefore V = -\frac{fR}{2} \pm \frac{fR}{2}\left(1 + \frac{2V_g}{fR}\right)$$

For this case, we are only interested in the positive root.

$$\begin{aligned}
&\therefore V = -\frac{fR}{2} + \frac{fR}{2} + V_g \\
&\therefore V = V_g
\end{aligned}$$

An alternative view on balanced flow

The gradient wind equation can be expressed in terms of the geostrophic wind:

$$\frac{V^2}{R} + fV - fV_g = 0, \ f < 0 \text{ in the Southern Hemisphere}$$

$$\implies \text{Centrifugal force} + \text{Coriolis force} + \text{pressure gradient force are in balance.}$$

The radius of curvature, R, can be obtained from this equation:

$$\frac{V^2}{R} = f(V_g - V)$$

$$\therefore R = \frac{V^2}{f(V_g - V)}$$

The balance of forces equation divided by the Coriolis force, fV, leads to

$$\frac{V^2}{fVR} + 1 - \frac{fV_g}{fV} = 0$$

$$\therefore \frac{V}{fR} + 1 - \frac{V_g}{V} = 0$$

$$\implies \frac{V}{fR} = \frac{V_g}{V} - 1$$

The last equation represents a straight line of the form $y = mx + c$, with $m = 1$ and $c = -1$: $y = x - 1$, with $y = \dfrac{V}{fR}$ and $x = \dfrac{V_g}{V}$.

Consider the following values of x and then calculate the corresponding y-values:

$$x \in \left\{-1, 0, \frac{1}{2}, 1, 2, 3\right\}.$$

<u>For $x = -1$</u>:

$$\frac{V_g}{V} = -1$$

$$\therefore V = -V_g, \ \text{defined here as anti-geostrophic flow.}$$

$$y = x - 1 = -1 - 1 = -2$$

$$R = \frac{V^2}{f(-V - V)} = -\frac{V}{2f} > 0, \ \text{anti-cyclonic flow}$$

<u>For $x = 0$</u>:

$$\frac{V_g}{V} = 0$$

$$\therefore V_g = 0$$

$$y = 0 - 1 = -1$$

$$R = \frac{V^2}{f(0 - V)} = -\frac{V}{f} > 0, \ \text{anti-cyclonic flow}$$

For $x = \frac{1}{2}$:

$$\begin{aligned} \frac{V_g}{V} &= \frac{1}{2} \\ \therefore V &= 2V_g \\ \therefore V &> V_g, \text{ anti-cyclonic flow} \\ y &= \frac{1}{2} - 1 = -\frac{1}{2} \\ R &= \frac{V^2}{f\left(\frac{1}{2}V - V\right)} = -\frac{2V}{f} > 0, \text{ anti-cyclonic flow} \end{aligned}$$

For $x = 1$:

$$\begin{aligned} \frac{V_g}{V} &= 1 \\ \therefore V &= V_g \\ y &= 1 - 1 = 0 \\ R &= \frac{V^2}{f(V - V)} \text{ is undefined.} \end{aligned}$$

For $x = 2$:

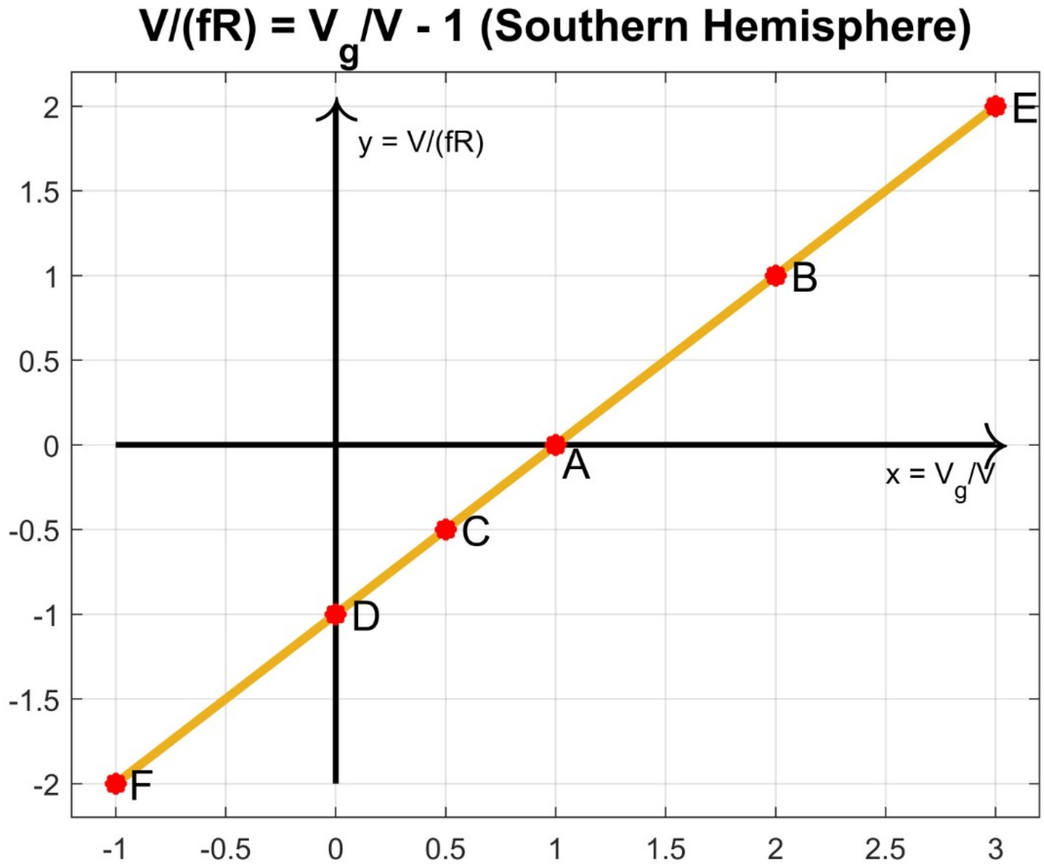
$$\begin{aligned} \frac{V_g}{V} &= 2 \\ \therefore V &= \frac{1}{2}V_g \\ \therefore V &< V_g, \text{ cyclonic flow} \\ y &= 2 - 1 = 1 \\ R &= \frac{V^2}{f\,(2V - V)} = \frac{V}{f} < 0, \text{ cyclonic flow} \end{aligned}$$

For $x = 3$:

$$\begin{aligned} \frac{V_g}{V} &= 3 \\ \therefore V &= \frac{1}{3}V_g \\ \therefore V &< V_g, \text{ cyclonic flow} \\ y &= 3 - 1 = 2 \\ R &= \frac{V^2}{f\,(3V - V)} = \frac{V}{2f} < 0, \text{ cyclonic flow} \end{aligned}$$

Also, $f = \frac{V}{2R} < \frac{V}{R}$ obtained from $x = 2$. Therefore, with x increasing, f decreases resulting in the Coriolis force becoming smaller.

We have now calculated the y-values of the straight line, resulting in the figure below:



For the x-value cases above, where $y < 0$, the flow has been found to be anti-cyclonic $\left(\text{e.g., } x = -1, 0, \frac{1}{2}\right)$. Conversely, where $y > 0$, the flow is cyclonic (e.g., $x = 2, 3$). Defining the flow to be sub-geostrophic where $V < V_g$ and super-geostrophic where $V > V_g$, sub-geostrophic flow is found where $x > 1$ and super-geostrophic for $0 < x < 1$. For $x < 0$, the flow is considered to be anti-geostrophic since $V = -V_g$.

Near point A ($x = 1, y = 0$), trajectories are nearly straight since $V \simeq V_g$ and $\frac{V_g}{V} \simeq 1$. However, since $\frac{V}{fR} = \frac{V_g}{V} - 1$, $\frac{V}{fR} \simeq 0$. Moreover, $\frac{V}{fR}$ is a Rossby number with R, the radius of curvature, the length scale. The smaller the Rossby number, the more dominant the Coriolis acceleration in the dynamics becomes, resulting in the Coriolis force and the Pgf becoming approximately balanced. So geostrophic balance is a good approximation near point A.

At point B ($x = 2, y = 1$), $R = \frac{V}{f}$, therefore $f = \frac{V}{R}$

$$\frac{V^2}{R} + fV - fV_g = 0 \text{ becomes } \frac{V^2}{R} + \frac{V}{R}V - fV_g = 0$$

Therefore, the centrifugal and Coriolis forces are equal and together balance the Pgf.

At point C $\left(x = \frac{1}{2}, y = -\frac{1}{2}\right)$, $R = -\frac{2V}{f}$ and $V_g = \frac{1}{2}V$.

$$\frac{V^2}{R} + fV - fV_g = 0 \text{ becomes } V^2\left(-\frac{f}{2V}\right) + fV - f\left(\frac{1}{2}V\right) = 0$$

$$\therefore -\frac{fV}{2} + fV - \frac{fV}{2} = 0$$

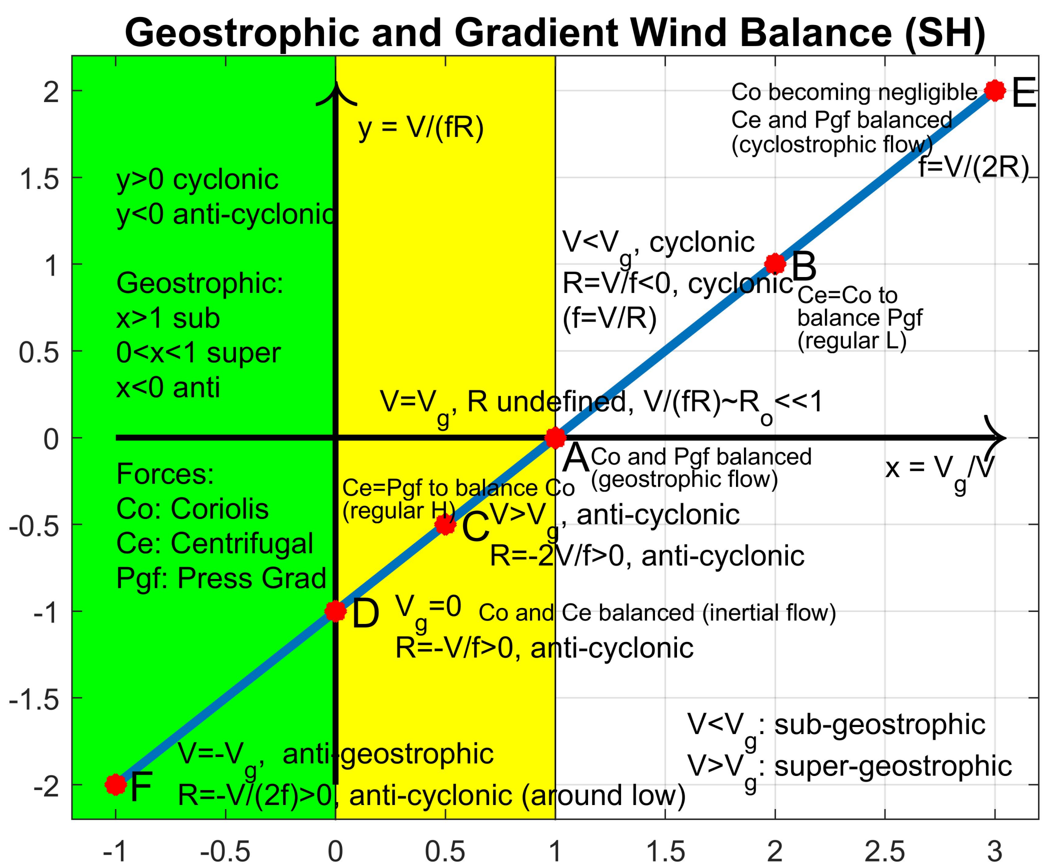
Here the centrifugal force and Pgf are equal and together balance the Coriolis force.

At point D $(x = 0, y = -1)$, $V_g = 0$, therefore the $Pgf (= -fV_g)$ is zero, and the Coriolis and centrifugal forces must balance.

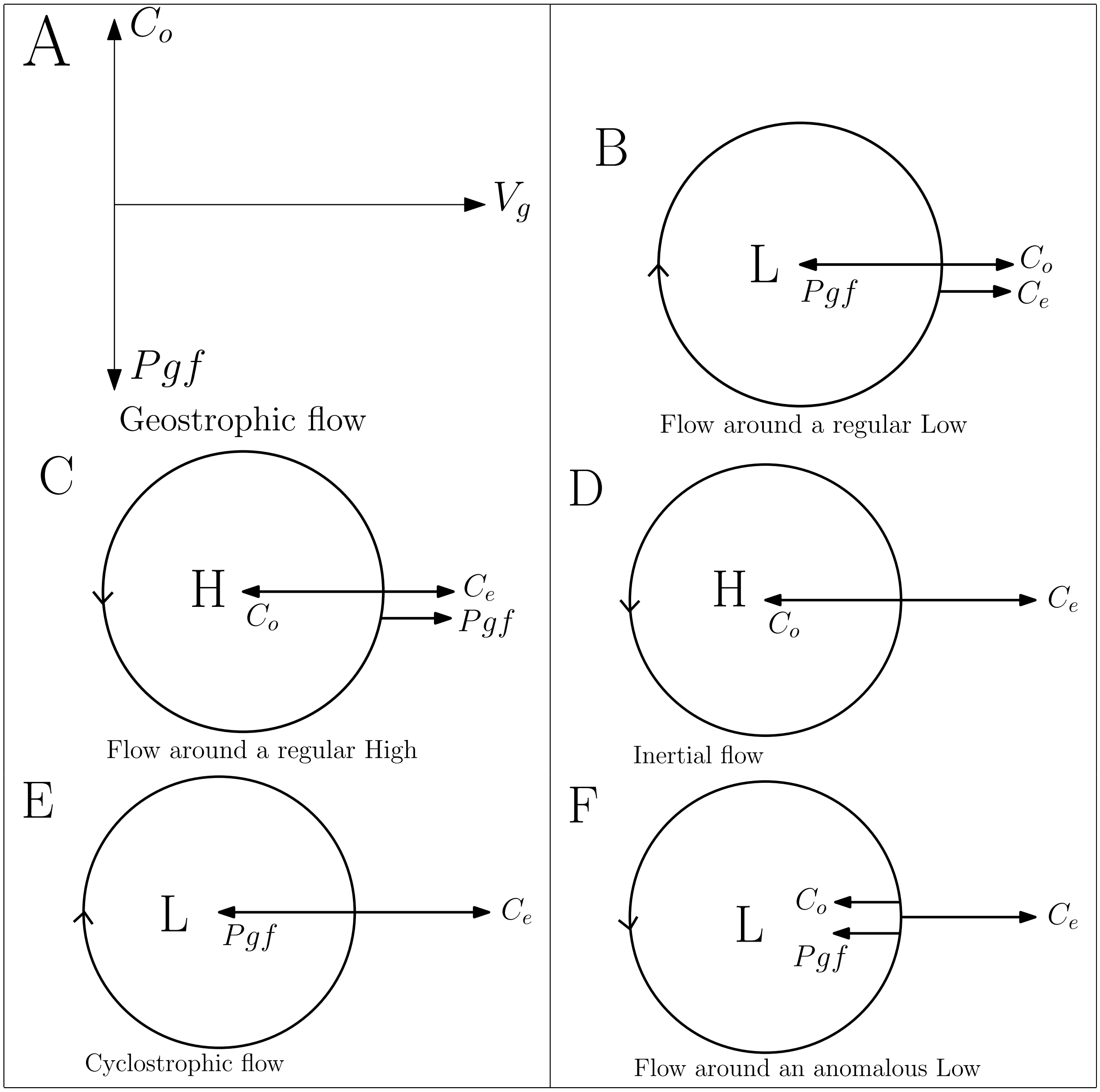
At point E $(x = 3, y = 2)$ and with an increasing x-value, the Coriolis force becomes negligible compared with the centrifugal force and Pgf. For this scenario, the centrifugal force must balance the Pgf alone, which is the definition of cyclostrophic balance.

Over the area on the graph where $x < 0$ (including point F), the flow is anti-geostrophic ($V = -V_g$) which means that the flow has anti-cyclonic curvature around a low-pressure system, i.e. an anomalous low, which is never observed.

The figure below summarizes these findings:



A summary of the force balances at each of the points (A-F) is given below:



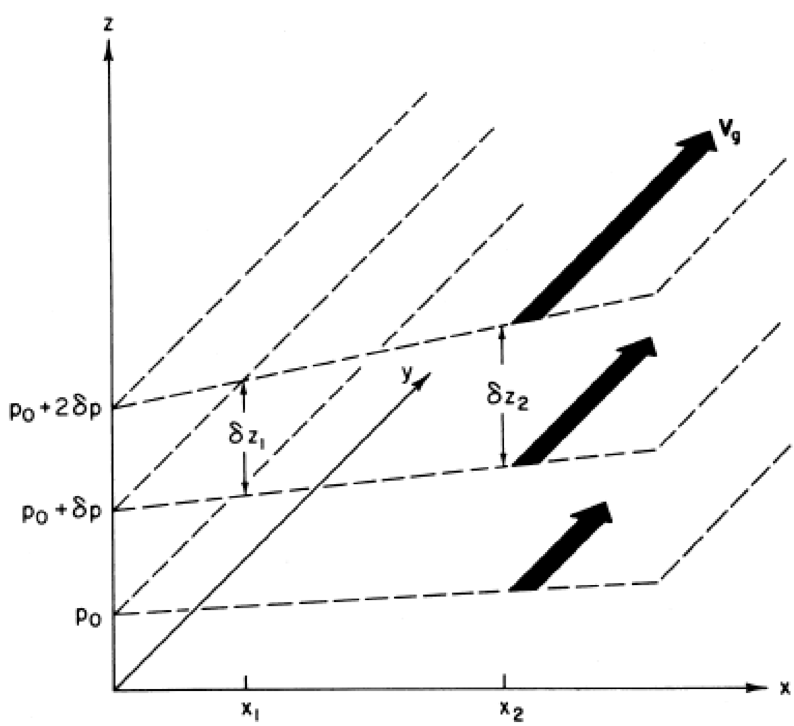
Variation with height of the geostrophic wind: The thermal wind

Isobaric coordinate form of the geostrophic relationship:

$$f\vec{V}_g = \vec{k} \times \vec{\nabla}_p \Phi \qquad \Phi : \text{geopotential}$$

$\Longrightarrow$ geostrophic wind $\propto$ geopotential gradient.

$\Longrightarrow$ geostrophic wind directed along the positive y-axis that increases in magnitude with height requires that the slope of the isobaric surfaces with respect to the x-axis must increase with height.



Hypsometric equation $\Phi(z_2) - \Phi(z_1) = g(z_2 - z_1) = R\int_{p_2}^{p_1} T d\ln p$

For thickness δz corresponding to a pressure interval δp

$$\delta z \approx -\frac{1}{g} RT \delta \ln p$$

$\implies$ thickness of the layer between isobaric surfaces $\propto$ temperature of the layer:

$$T(\delta z_1) < T(\delta z_2)$$

$\implies$ increase of height of a positive x-directed pressure gradient is associated with a positive x-directed temperature gradient

$\implies$ the air in a vertical column at x_2, because it is warmer (less dense), must occupy a greater depth for a given pressure drop than the air at x_1

From $\vec{V_g} = \frac{1}{f} \vec{k} \times \vec{\nabla}_p \Phi$, in isobaric coordinates:

$$v_g = \frac{1}{f} \frac{\partial \Phi}{\partial x} \text{ and } u_g = -\frac{1}{f} \frac{\partial \Phi}{\partial y}$$

Equation of state for an ideal gas: $p\alpha = RT$ or $p = \rho RT$ (1.17)
where $\alpha = \rho^{-1}$

Geopotential: $\delta \Phi = g \delta z \implies \delta z = \frac{1}{g} \delta \Phi$

Hydrostatic equation: $\frac{\delta p}{\delta z} = -\rho g \implies \frac{\delta z}{\delta p} = -\frac{1}{\rho g}$

$$\begin{aligned} \therefore \frac{1}{g} \frac{\delta \Phi}{\delta p} &= -\frac{\alpha}{g} \\ \lim_{\delta p \to 0} \frac{\delta \Phi}{\delta p} = \frac{\partial \Phi}{\partial p} &= -\alpha \\ &= -\frac{RT}{p} \qquad (3.27) \\ \therefore T &= -\frac{p}{R} \frac{\partial \Phi}{\partial p} \end{aligned}$$

Differentiate geostrophic wind components with respect to pressure:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right) = \frac{1}{f}\frac{\partial}{\partial x}\left(-\frac{RT}{p}\right)$$

$$\therefore p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p$$

$$\therefore \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right) = -\frac{1}{f}\frac{\partial}{\partial y}\left(-\frac{RT}{p}\right)$$

$$\therefore \frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \tag{3.29}$$

As a vector:

$$\underbrace{\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R}{f}\vec{k} \times \vec{\nabla}_p T}_{\text{the thermal wind equation}} \tag{3.30}$$

Here we have shown that the geostrophic wind must have vertical shear in the presence of a horizontal temperature gradient.

$$\vec{V}_T \equiv \vec{V}_g(p_1) - \vec{V}_g(p_0)$$

$$\therefore \vec{V}_T = -\frac{R}{f}\int_{p_0}^{p_1} \vec{k} \times \vec{\nabla}_p T d\ln p \tag{3.31}$$

$\langle T \rangle$ is the mean temperature in the layer between p_0 and p_1.

$$u_T = +\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \int_{p_0}^{p_1} d\ln p$$

$$= -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p [-\ln(p_1) + \ln(p_0)]$$

$$= -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right) \tag{3.32}$$

$$\text{and } v_T = -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)_p \int_{p_0}^{p_1} d\ln p$$

$$= \frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right) \tag{3.32}$$

Thermal wind is the vector difference between geostrophic winds at two levels:

$$\vec{V}_T \equiv \vec{V}_g(p_1) - \vec{V}_g(p_0) \qquad (p_1 < p_0)$$

$$\text{also } u_T = u_{g_1} - u_{g_0} = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1) - \left(-\frac{1}{f}\frac{\partial}{\partial y}(\Phi_0)\right)$$

$$= -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \tag{3.33}$$

$$v_T = v_{g_1} - v_{g_0} = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1) - \frac{1}{f}\frac{\partial}{\partial x}(\Phi_0)$$

$$= \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \tag{3.33}$$

$$\implies -\frac{R}{f}\left(\frac{\partial \langle T\rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0)$$

$$\implies R\ln\left(\frac{p_0}{p_1}\right)\int \frac{\partial \langle T\rangle}{\partial y}dy = \int \frac{\partial}{\partial y}(\Phi_1 - \Phi_0)dy$$

$$\implies \Phi_1 - \Phi_0 = R\langle T\rangle \ln\left(\frac{p_0}{p_1}\right) \tag{3.34}$$

Per definition: $\Phi_1 - \Phi_0 \equiv Z_T g$, where Z_T is the **thickness**

Also

$$\frac{R}{f}\left(\frac{\partial \langle T\rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0)$$

$$R\ln\left(\frac{p_0}{p_1}\right)\int \frac{\partial \langle T\rangle}{\partial x}dx = \int \frac{\partial}{\partial x}(\Phi_1 - \Phi_0)dx$$

$$\implies \Phi_1 - \Phi_0 = R\langle T\rangle \ln\left(\frac{p_0}{p_1}\right) \tag{3.34}$$

The thickness is therefore proportional to the mean temperature in the layer.

$\implies$ lines of **equal thickness** are equivalent to the **isotherms** of mean temperature in the layer.

(3.35):

$$\vec{V}_T = \underbrace{\frac{1}{f}\vec{k}\times\vec{\nabla}(\Phi_1 - \Phi_0)}_{\text{From (3.33)}} = \underbrace{\frac{g}{f}\vec{k}\times\vec{\nabla}Z_T}_{\text{From (3.33),(3.34)}} = \underbrace{\frac{R}{f}\vec{k}\times\vec{\nabla}\langle T\rangle \ln\left(\frac{p_0}{p_1}\right)}_{\text{From (3.32)}}$$

Exercise 1: The mean temperature in the layer between 750 and 500 hPa **decreases eastward** by 2°C per 100 km. If the 750 hPa geostrophic wind is from the southeast at $20\,\mathrm{m\,s^{-1}}$, what is the geostrophic wind speed at 500 hPa? Let $f = -10^{-4}\,\mathrm{s}^{-1}$ [Hint: remember Pythagoras when calculating the geostrophic wind components].

Solution: The mean temperature decreases eastward, so there is no north–south component: $\frac{\partial \langle T\rangle}{\partial y} = 0$ and $\frac{\partial \langle T\rangle}{\partial x} < 0$

$$\therefore u_T = 0$$

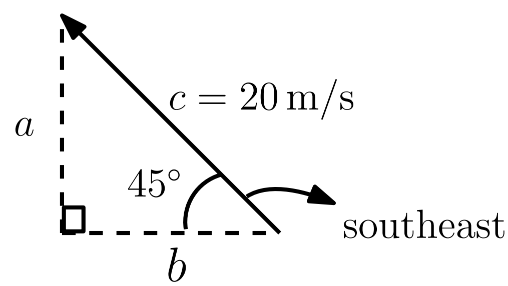
$$v_T = \frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)\ln\left(\frac{p_0}{p_1}\right)$$

where R is the gas constant for dry air.

$$\begin{aligned} R &= 287\,\mathrm{J\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{N\,m\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{kg\,m\,s^{-2}\,m\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{m^2\,s^{-2}\,K^{-1}} \end{aligned}$$

$$\begin{aligned} \therefore v_T &= \frac{287\,\mathrm{m^2\,s^{-2}\,K^{-1}}}{-10^{-4}\,\mathrm{s^{-1}}}\left(-\frac{2\,\mathrm{K}}{100\,000\,\mathrm{m}}\right)\ln\left(\frac{750}{500}\right) \\ &= 23.27\,\mathrm{m\,s^{-1}} \end{aligned}$$

At $750\,\mathrm{hPa}$:



$$\begin{aligned} c^2 = a^2 + b^2 = 2a^2 \implies a &= \left(\frac{20^2}{2}\right)^{1/2} = b \\ &= 14.14\,\mathrm{m\,s^{-1}} \end{aligned}$$

$$\begin{aligned} u_T = u_{g_1} - u_{g_0} &= u_g(500\,\mathrm{hPa}) - u_g(750\,\mathrm{hPa}) \\ \therefore 0 &= u_g(500) - (-14.14) \\ \therefore u_g(500) &= -14.14\,\mathrm{m\,s^{-1}} \end{aligned}$$

$$\begin{aligned} v_T = v_{g_1} - v_{g_0} &= v_g(500\,\mathrm{hPa}) - v_g(750\,\mathrm{hPa}) \\ \therefore 23.27 &= v_g(500) - 14.14 \\ \therefore v_g(500) &= 37.41\,\mathrm{m\,s^{-1}} \end{aligned}$$

$$\therefore \vec{V}_g(500) = (-14.14, 37.41)$$

$$\begin{aligned} \text{Therefore, geostrophic wind speed at } 500\,\mathrm{hPa} &= \left|\vec{V}_g(500)\right| \\ &= \left((-14.14)^2 + (37.41)^2\right)^{1/2} \\ &= 39.99\,\mathrm{m\,s^{-1}} \end{aligned}$$

Exercise 2: Consider the values in the previous exercise (Exercise 1) above, what is the mean temperature advection in the 750 to 500 hPa layer?

Solution: Only west–east component: $\vec{V} \cdot \vec{\nabla} T = u \dfrac{\partial T}{\partial x}, \quad \dfrac{\partial T}{\partial x} = -\dfrac{2°\text{C}}{100\,\mathrm{km}}$

But temperature in the layer is decreasing $\implies u\dfrac{\partial T}{\partial x} < 0$

Since we are considering a layer, we use mean values:

$$\text{Temperature advection in the layer} = -\overline{u}\frac{\partial \overline{T}}{\partial x}$$

where bars denote the means.

$$\overline{u} = \frac{(u_g(500) + u_g(750))}{2} = \frac{(-14.14 - 14.14)}{2} = -14.14\,\mathrm{m\,s^{-1}}$$

$$\begin{aligned} \therefore \overline{u}\frac{\partial \overline{T}}{\partial x} &= (-14.14)\left(-\frac{2}{100\,000}\right)\mathrm{m\,s^{-1}\,K\,m^{-1}} \\ &= 2.828 \times 10^{-4}\,\mathrm{K\,s^{-1}} \qquad (\times 3600) \\ &= 1.018\,\mathrm{K\,h^{-1}} \qquad (°\mathrm{C\,h^{-1}}) \end{aligned}$$

$$\text{Therefore, temperature advection in the layer} = -\overline{u}\frac{\partial \overline{T}}{\partial x} = -1.018\,\mathrm{K\,h^{-1}}$$

Bonus Homework: Describe the relationship between turning of geostrophic wind and temperature advection in terms of backing and veering of the wind with height for the Southern Hemisphere.

Barotropic and baroclinic atmospheres

Barotropic atmosphere: $\rho = \rho(p)$; thermal wind equation $\dfrac{\partial \vec{V}_g}{\partial \ln p} = 0$, which states that the geostrophic wind is independent of height.

Baroclinic atmosphere: $\rho = \rho(p, T)$; geostrophic wind has vertical sheer, related to the horizontal temperature gradient.

Vertical motion

In general the vertical velocity component of synoptic-scale motions is not measured directly, but must be inferred from fields that are measured directly.

Two commonly used methods for inferring vertical motion:

1. Kinematic (based on continuity equation)
2. Adiabatic (based on thermodynamic energy equation)

$$\begin{aligned}
\omega &= \omega(p) \ \text{ vertical velocity in isobaric coordinates.} \\
\omega &\equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \\
&= \frac{\partial p}{\partial t} + \vec{V} \cdot \vec{\nabla} p + w\frac{\partial p}{\partial z} \\
&= \frac{\partial p}{\partial t} + (\vec{V}_g + \vec{V}_a) \cdot \vec{\nabla} p - g\rho w \qquad \left(\text{Since } \frac{\partial p}{\partial z} = -\rho g\right)
\end{aligned}$$

$\vec{V}_a$: Ageostrophic wind, $|\vec{V}_a| \ll |\vec{V}_g|$ the geostrophic wind

$$\therefore \omega = \frac{\partial p}{\partial t} + \vec{V}_g \cdot \vec{\nabla} p + \vec{V}_a \cdot \vec{\nabla} p - g\rho w$$

Bonus Homework: Show that $\vec{V}_g \cdot \vec{\nabla} p = 0 \qquad \left(\vec{V}_g = \frac{1}{\rho f}\vec{k} \times \vec{\nabla} p\right)$

$$\therefore \omega = \frac{\partial p}{\partial t} + \vec{V}_a \cdot \vec{\nabla} p - g\rho w \tag{3.37}$$

Scale analysis:

$$\begin{aligned}
\frac{\partial p}{\partial t} &\sim 10\,\text{hPa}\,/\,\text{day} \qquad [1\,\text{hPa} = 100\,\text{Pa}] \\
\vec{V}_a \cdot \vec{\nabla} p &\sim (1\,\text{m}\cdot\text{s}^{-1})(0.01\,\text{hPa}\,/\,\text{km}) \sim 1\,\text{hPa}\,/\,\text{day} \\
g\rho w &\sim 100\,\text{hPa}\,/\,\text{day}
\end{aligned}$$

Therefore, a good approximation is

$$\omega = -g\rho w \tag{3.38}$$

Kinematic method

One method of deducing the vertical velocity. Integration of the continuity equation $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p+\frac{\partial \omega}{\partial p}=0$ with respect to pressure from a reference level p_s to any level p, yields

$$w(z)=\frac{\rho(z_s)w(z_s)}{\rho(z)}-\frac{p_s-p}{\rho(z)g}\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right), \tag{3.40}$$

where z and z_s are the heights of pressure levels p and p_s, respectively.

Derivation of (3.40):

(3.5): $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p+\frac{\partial \omega}{\partial p}=0 \implies \frac{\partial \omega}{\partial p}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

Integrate this expression with respect to pressure from a reference level p_s to any level p:

$$\int_{p_s}^{p}\frac{\partial \omega}{\partial p}dp=-\int_{p_s}^{p}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dp$$
$$\therefore \omega(p)-\omega(p_s)=-\left[\frac{\partial}{\partial x}\int_{p_s}^{p}udp+\frac{\partial}{\partial y}\int_{p_s}^{p}vdp\right]$$

Define a pressure weighted vertical average:

$$\langle A\rangle \equiv (p-p_s)^{-1}\int_{p_s}^{p}Adp$$
$$\therefore \int_{p_s}^{p}Adp=(p-p_s)\langle A\rangle=-(p_s-p)\langle A\rangle$$

$$\therefore \omega(p)-\omega(p_s)=-\left[\frac{\partial}{\partial x}\{-(p_s-p)\langle u\rangle\}+\frac{\partial}{\partial y}\{-(p_s-p)\langle v\rangle\}\right]$$
$$=(p_s-p)\frac{\partial\langle u\rangle}{\partial x}+(p_s-p)\frac{\partial\langle v\rangle}{\partial y}$$
$$\therefore \omega(p)=\omega(p_s)+(p_s-p)\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right)$$

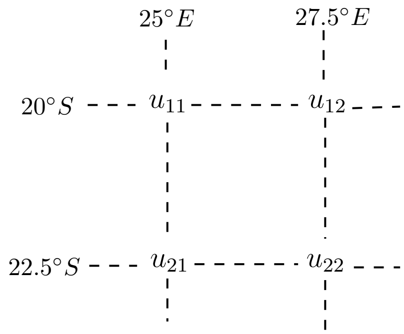
Since $\omega=-\rho g w$, we get $-\rho(z)gw(z)=-\rho(z_s)gw(z_s)+(p_s-p)\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right)$

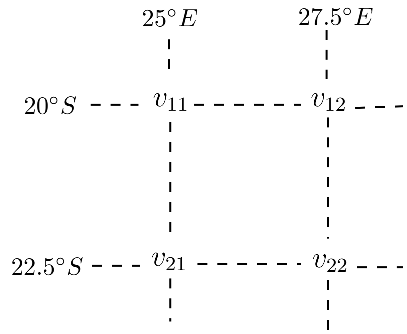
$$w(z)=\frac{\rho(z_s)w(z_s)}{\rho(z)}-\frac{p_s-p}{\rho(z)g}\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right)$$

To infer the vertical velocity from the equation above requires knowledge of the horizontal divergence:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

Consider the table below showing the u and v components of the wind ($\mathrm{m\cdot s^{-1}}$) for the $200\,\mathrm{hPa}$ level, on a 2.5° lat–long grid:





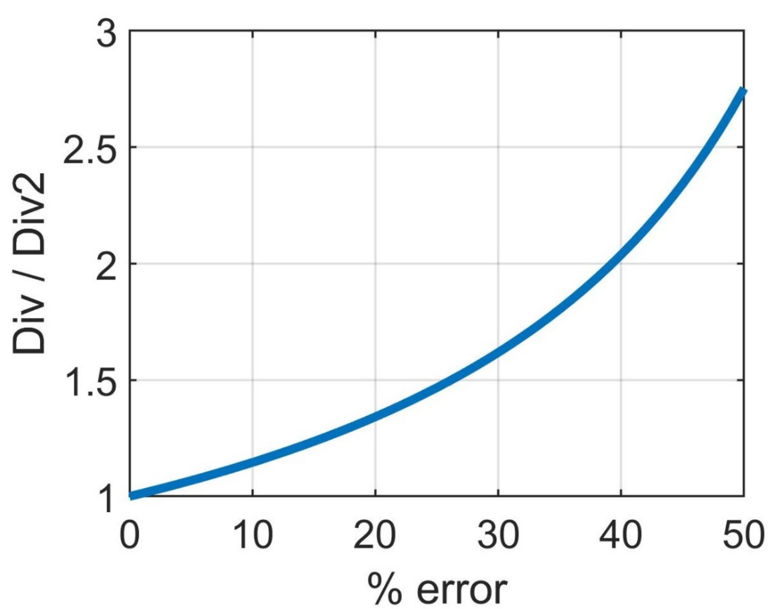
<u>Using finite difference approximations</u>:

$$\frac{\delta u}{\delta x} = [(u_{12} + u_{22})/2 - (u_{11} + u_{21})/2]/\delta x$$
$$\frac{\delta v}{\delta y} = [(v_{11} + v_{12})/2 - (v_{21} + v_{22})/2]/\delta y$$
$$\delta y = 2\pi 6.37 \times 10^6/144; \quad \delta x = \delta y \cos(21.25^\circ)$$

Typical values for u and v:

$$u = [28.8 \quad 28.7; \quad 34.6 \quad 37.4]$$
$$v = [-20.5 \quad -18.2; \quad -26.9 \quad -25.0]$$

An error in one of the wind components can lead to an exponential growth in the estimated divergence. See figure below. For this reason, the continuity equation method is not recommended for estimating the vertical motion field from observed horizontal winds.



Adiabatic method

The **adiabatic method** for inferring vertical velocities, which is not so sensitive to **errors** in the measured horizontal velocities, is based on the thermodynamic energy equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \tag{3.6}$$

If J, the diabatic heating, is small:

$$\begin{aligned}\omega &= \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)\\ &= \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\vec{\nabla}T\right)\end{aligned}$$

The temperature advection $\vec{V}\cdot\vec{\nabla}T$ can be accurately obtained from geostrophic winds, and so this method can be applied.

However, $\dfrac{\partial T}{\partial t}$ is difficult to estimate accurately since observations are not typically at close time intervals.

This method is also inaccurate when J is not small (i.e., strong diabatic heating) as is the case of storms in which heavy rainfall occurs over a large area.

Exercise: For a high altitude station located near the 750 to $500\,\mathrm{hPa}$ layer, the temperature is decreasing at a rate of 2°C per hour. Compute the vertical velocity in $\mathrm{cm/s}$ using the adiabatic method. Suppose the lapse rate at the station is $4°\mathrm{C}/\,\mathrm{km}$, temperature advection is $-2.828\times10^{-4}\,\mathrm{K\,s^{-1}}$, and that the dry adiabatic lapse rate is determined by gravity and by the specific heat of dry air at constant pressure.

Solution: $\dfrac{\partial T}{\partial t} = -2°\mathrm{C\,h^{-1}}$ (decreasing)

$$\text{Adiabatic method: } \omega = \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\vec{\nabla}T\right), \quad S_p = \frac{\Gamma_d - \Gamma}{\rho g} \text{ and } \omega = -\rho g w$$

$$\therefore w = \frac{\left(\dfrac{\partial T}{\partial t} + \vec{V}\cdot\vec{\nabla}T\right)}{\Gamma - \Gamma_d}$$

$$\begin{aligned}\Gamma_d = \frac{g}{c_p} = \frac{9.81\,\mathrm{m\,s^{-2}}}{1005\,\mathrm{J\,K^{-1}\,kg^{-1}}} &= 9.771\times10^{-3}\,\mathrm{m\,s^{-2}}(\mathrm{J\,K^{-1}\,kg^{-1}})^{-1}\\ &= 9.771\times10^{-3}\,\mathrm{m\,s^{-2}}(\mathrm{kg\,m\,s^{-2}\,m\,K^{-1}\,kg^{-1}})^{-1}\\ &= 9.771\times10^{-3}\,\mathrm{K\,m^{-1}}\end{aligned}$$

$$\Gamma = 4\,\mathrm{K\,m^{-1}} = 4\times10^{-3}\,\mathrm{K\,m^{-1}}$$

$$
\begin{aligned}
w &= \frac{\left(\left(-\frac{2}{3600}\right)\mathrm{K\,s^{-1}} - 2.828\times10^{-4}\,\mathrm{K\,s^{-1}}\right)}{(4\times10^{-3} - 9.771\times10^{-3})\,\mathrm{K\,m^{-1}}} \\
&= 0.1453\,\mathrm{m\,s^{-1}} \\
&= 14.53\,\mathrm{cm\,s^{-1}}
\end{aligned}
$$

Surface pressure tendency

The development of a **negative surface pressure tendency** is a classic warning of an **approaching cyclonic weather disturbance**.

$$\omega(p) = \omega(p_s) - \int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp \tag{3.39}$$

$$\text{where } \lim_{p \to 0} \implies 0 = \omega(p_s) + \int_{0}^{p_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp$$

$$\therefore \omega(p_s) = - \int_{0}^{p_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp$$

$$= - \int_{0}^{p_s} (\vec{\nabla} \cdot \vec{V}) dp \tag{3.43}$$

$$\omega = \frac{\partial p}{\partial t} + \vec{V}_a \cdot \vec{\nabla} p - g\rho w \tag{3.37}$$

Assumption: w at the surface $= 0$, and $\vec{V}_a \cdot \vec{\nabla} p$ can be neglected (scaling considerations)

$$\therefore \omega \approx \frac{\partial p}{\partial t}$$

$$\therefore \frac{\partial p}{\partial t} \approx - \int_{0}^{p_s} (\vec{\nabla} \cdot \vec{V}) dp \tag{3.44}$$

In words: The surface **pressure tendency** at a given point is determined by the total convergence (negative divergence) of mass into the vertical column of atmosphere above that point.

The utility of the tendency equation is severely limited due to the fact that $\vec{\nabla} \cdot \vec{V}$ is difficult to compute from observations because it depends on the ageostrophic wind field.

Bonus Homework: Describe qualitatively the origin of surface pressure changes and the relationship of such changes to the horizontal divergence.

The circulation theorem

Circulation about a closed contour in a fluid:

$$
\begin{aligned}
C &\equiv \oint \vec{U} \cdot d\vec{l} \qquad (d\vec{l} \text{ is the displacement vector locally tangent to the contour}) \\
&= 2\Omega\pi R^2 \qquad R\text{: radius of circular ring of fluid}
\end{aligned}
$$

$\Longrightarrow$ the circulation is 2π times the angular momentum of the fluid.

By integrating Newton's second law, we can obtain the circulation theorem in an absolute coordinate system as:

$$\frac{DC_a}{Dt} = \frac{D}{Dt}\oint \vec{U}_a \cdot d\vec{l} = -\oint \frac{1}{\rho} dp$$

The solenoidal term is $-\oint \frac{1}{\rho} dp$

In meteorological analysis it is more convenient to work with the relative circulations C.

$$
\begin{aligned}
C &= C_a - C_e \qquad [C_e \text{ : due to Earth's rotation}] \\
&= C_a - 2\Omega A_e \qquad [A \text{ : area}] \\
\frac{DC}{Dt} &= \frac{DC_a}{Dt} - 2\Omega\frac{DA_e}{Dt} \\
&= -\oint \frac{1}{\rho} dp - 2\Omega\frac{DA_e}{Dt}
\end{aligned}
$$

Vorticity

Definition: The microscopic measure of rotation in a fluid. It is a vector field defined as the curl of velocity.

$$\text{Absolute vorticity} \;\; \omega_a \equiv \vec{\nabla} \times \vec{U}_a$$

$$\text{Relative vorticity} \;\; \omega \equiv \vec{\nabla} \times \vec{U} \qquad (\vec{U} \text{ is the relative velocity})$$

$$\begin{aligned}
\therefore \omega &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{i}u + \vec{j}v + \vec{k}w) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \vec{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

For large-scale dynamic meteorology, the concern is only with the vertical components of absolute and relative vorticity.

$$\text{Absolute vorticity} \;\; \eta \equiv \vec{k} \cdot (\vec{\nabla} \times \vec{U}_a)$$

$$\text{Relative vorticity} \;\; \zeta \equiv \vec{k} \cdot (\vec{\nabla} \times \vec{U})$$

Regions of $\zeta < 0$ are associated with cyclonic storms in the Southern Hemisphere.

The distribution of ζ is an excellent diagnostic for weather analysis.

Planetary vorticity: the local vertical component of the vorticity of the earth due to its rotation

$$\vec{k} \cdot \vec{\nabla} \times \vec{U}_e = 2\Omega \sin\phi = f, \text{ the Coriolis parameter}$$

$$\eta = \zeta + f$$

$$\begin{aligned}
\zeta &= \vec{k} \cdot \left(\vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) \\
&= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
\therefore \eta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f
\end{aligned}$$

Exercise 1: What is the relative vorticity on the side of a current which decreases in magnitude towards the south at a rate of $10\,\mathrm{m}\,/\,\mathrm{s}$ for every $500\,\mathrm{km}$?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

No west–east component: $\dfrac{\partial v}{\partial x} = 0$

$$\begin{aligned} \frac{\partial u}{\partial y} &< 0 \qquad \text{(towards the south)} \\ \therefore \frac{\partial u}{\partial y} &= -\frac{10\,\mathrm{m\,s^{-1}}}{500\ 000\,\mathrm{m}} = -2 \times 10^{-5}\,\mathrm{s^{-1}} \\ \therefore \zeta &= 0 - (-2 \times 10^{-5}\,\mathrm{s^{-1}}) \\ &= 2 \times 10^{-5}\,\mathrm{s^{-1}} \end{aligned}$$

Exercise 2: An air parcel at 30°S moves southward conserving absolute vorticity (the initial absolute vorticity is equal to the final absolute vorticity). If its initial relative vorticity is $5 \times 10^{-5}\,\mathrm{s^{-1}}$, what is its relative vorticity upon reaching 90°S?

Solution:

$$(\zeta + f)_{\text{initial}} = (\zeta + f)_{\text{final}}$$

$$\begin{aligned} f_{\text{initial}} &= 2\Omega \sin(-30°) = 2\Omega \left(-\frac{1}{2}\right) = -\Omega \\ f_{\text{final}} &= 2\Omega \sin(-90°) = 2\Omega\,(-1) = -2\Omega \\ \zeta_{\text{initial}} &= 5 \times 10^{-5}\,\mathrm{s^{-1}} \quad \text{(given)} \end{aligned}$$

$$\begin{aligned} \zeta_{\text{final}} &= (\zeta + f)_{\text{initial}} - f_{\text{final}} \\ &= 5 \times 10^{-5} - \Omega - (-2\Omega) \\ &= 5 \times 10^{-5} + \Omega \\ &= 5 \times 10^{-5} + 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}} \\ &= 12.292 \times 10^{-5}\,\mathrm{s^{-1}} \end{aligned}$$

<u>Bonus Homework</u>: Determine the relationship between relative vorticity and relative circulation (macroscopic).

Potential vorticity

Definition and characteristics of potential vorticity

The potential vorticity (PV) is the absolute circulation of an air parcel that is enclosed between two isentropic surfaces (a surface in space on which potential temperature is everywhere equal). If PV is displayed on a surface of constant potential temperature, then it is officially called IPV (isentropic potential vorticity). PV could also be displayed on another surface, for example a pressure surface. Note from the relation below, that PV is simply the product of absolute vorticity on an isentropic surface and static stability. So PV consists, in contrast to vorticity on isobaric surfaces, of two factors, a dynamical element and a thermodynamical element.

$$PV \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

where,

f is the Coriolis parameter

g is the gravitational acceleration

p is the pressure

PV is the potential vorticity

θ is the potential temperature: $\theta = T\left(\frac{p_s}{p}\right)^{R/c_p}$

ζ_θ is the relative isentropic vorticity [the vertical component of relative vorticity evaluated on an isentropic surface]

Within the troposphere, the values of PV are usually low. However, the potential vorticity increases rapidly from the troposphere to the stratosphere due to the significant change of the static stability. Typical changes of the potential vorticity within the area of the tropopause are from 1 (tropospheric air) to 4 (stratospheric air) PV units (PV unit: $1\,\mathrm{PVU} = 10^{-6}\,\mathrm{K\,kg^{-1}\,m^2\,s^{-1}}$). Today in most of the literature the 2 PV unit anomaly, which separates tropospheric from stratospheric air, is referred to as dynamical tropopause. The traditional way of describing the tropopause, is with use of the potential temperature or static stability. This is only a thermodynamical way of characterising the tropopause. The benefit of using PV is that the tropopause can be understood in both thermodynamic and dynamic terms. An abrupt folding or lowering of the dynamical tropopause can also be called an upper PV-anomaly. When this occurs, stratospheric air penetrates into the troposphere resulting in high values of PV with respect to the surroundings, creating a positive PV-anomaly.

In the lower levels of the troposphere, strong baroclinic zones often occur which can be regarded as low level PV anomalies.

It must be stressed that this other way of looking at the dynamics of the atmosphere will not necessarily result in new conclusions. However, it may give new dimensions to things that, in fact, were already known.

The two main advantages of potential vorticity (with certain assumptions) are: conservation and invertibility. The two advantages will be discussed briefly:

Conservation

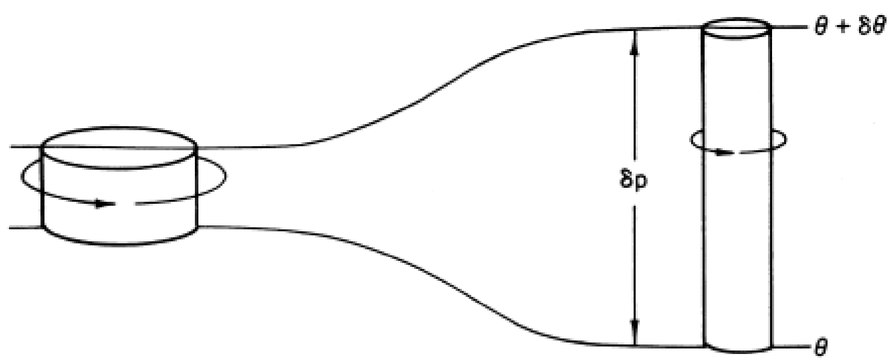
With the following assumptions PV is a conserved parameter:

1. Adiabatic stream (no diabatic heating or cooling)
2. No friction
3. Homogenous
4. Non-compressing

A first mathematical consequence of the conservation can be derived from the definition of PV: A parcel will keep the same value of PV if it moves along an adiabat through the atmosphere thus the equation for PV can be written as:

$$PV \equiv (\zeta_\theta + f)\left(-g\frac{\partial \theta}{\partial p}\right) = \text{constant} \tag{4.12}$$

Due to the conservation of PV, there is a close relationship between absolute vorticity and static stability (the ability of a fluid at rest to become turbulent or laminar [flow taking place along constant streamlines, without turbulence] due to the effects of buoyancy). The diagram below shows a parcel (cylinder) that is confined between potential temperature (isentropic) surfaces θ and $\theta + \delta\theta$ which are separated by a pressure interval δp. Difference in potential temperature between the top and bottom is the same for the two cylinders. If PV is conserved, and the cylinder is stretched, then static stability is decreasing and absolute vorticity must increase. Alternatively, if one goes from the stretched cylinder to the squashed cylinder, then static stability is increasing and absolute vorticity must decrease.



Due to the conservation of PV, significant features that are related to synoptic scale weather systems can be identified and followed in space as well as in time. This is a very powerful characteristic of this property.

Especially the case of a lowering of the dynamical tropopause, the upper PV-anomaly can be followed in time and space rather easily. PV anomalies are well related to a lot of dynamical processes in the troposphere. A distinct example of this are cases of Rapid Cyclogenesis where PV-anomalies play an important role.

The sudden creation or destruction of PV means that diabatic processes are involved (release of latent heat, friction, radiation). This fact can be used as tool to identify or even quantify the influence of these processes.

Invertibility

The second advantage of PV, invertibility, is a very important tool, because it allows one to obtain familiar meteorological fields, like the geopotential, wind, temperature and the static stability, when the distribution of the PV and the boundary conditions, potential temperature at the surface, are known. Further with the help of the invertibility it is possible to quantify the importance of PV-anomalies and the strength of their associated circulation and/or temperature pattern.

Inverting the PV for the entire atmosphere is interesting, but a more insightful diagnostic technique is piecewise PV inversion (PPVI). This involves dividing the atmosphere into significant layers and independently inverting the PV in those layers. This technique allows for analysis of the influence of discreet portions of the total PV field on the flow throughout the domain.

PV-thinking in the real atmosphere

The dynamical tropopause

The tropopause separates the well-mixed troposphere with the highly stratified, statically stable stratosphere. The tropopause is conventionally thought of from a thermal point of view and is based on the vertical temperature lapse rate. However, since high-PV values are generally associated with highly statically stable air, the tropopause can also be defined by the isentropic (contours of constant potential temperature) gradient of PV. The PV definition of the troposphere is known as the dynamical tropopause. By convention, the dynamical tropopause is usually defined by a constant PV contour which separates tightly packed PV contours of the stratosphere and low vertical gradient PV contours of the troposphere. A value between -1.5 and -2.5 PVU is most commonly used.

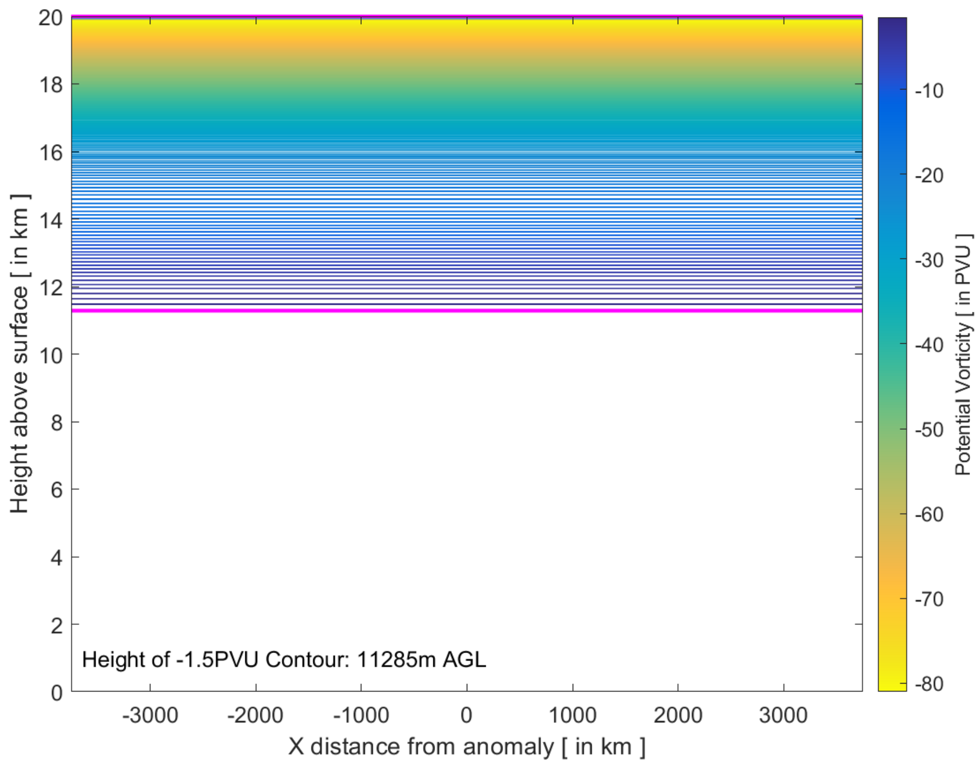


Figure 8: Cross-sectional PV in an idealised, mean atmosphere.

PV anomalies

Mathematically, an anomaly is the departure of a value from the mean distribution. A high-PV anomaly will thus be where there are anomalously high values (large negative values in the southern hemisphere) of PV compared to the mean distribution. Conversely, a low-PV anomaly will have anomalously small values (smaller negative values) of PV compared to the mean distribution.

Upper level PV anomalies

As seen in Fig. 8, there exists a reservoir of high-PV air in the stratosphere. Thus, stratospheric air is a source of high-PV anomalies in the troposphere. Upper-level high-PV anomalies can therefore be viewed, from a cross-sectional point of view, as tongues of high-PV stratospheric air intruding into the troposphere towards the surface. An idealised example of this is shown in Fig. 9. We recall that the circulation can be inferred from the PV distribution by the power of PV inversion and recall that PV can be represented by equation (4.12). The high-PV (negative PV anomaly) induces a negative vorticity anomaly. Flow is cyclonic around a negative vorticity anomaly in the Southern Hemisphere and hence cyclonic around the high-PV anomaly. Since the atmosphere is in thermal wind balance, the velocity of the circulation above and below the anomaly will also be cyclonic but the flow will be weaker.

The fact that the atmosphere is in thermal wind balance allows for us to decipher the temperature structure of the sectors. Recalling that the definition of the thermal wind is the difference between the upper and lower wind vectors ($\vec{V}_T = \vec{V}_g(p_1) - \vec{V}_g(p_0)$) and that the cold pool lies to the left (right) of the thermal wind

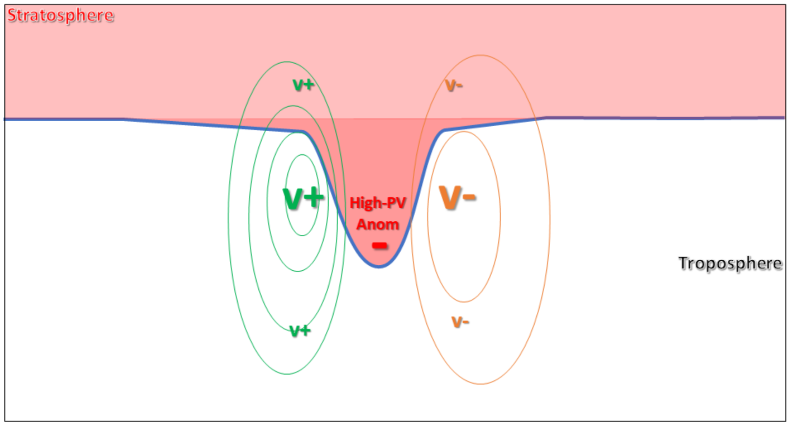


Figure 9: A cross-sectional view of an idealised PV intrusion in the upper troposphere inducing upper-level cyclonic flow around it.

vector in the Northern (Southern) hemisphere, it follows that there must exist a cold pool below the high-PV anomaly. Similarly, there must exist a warm pool in the stratospheric sector above the high-PV anomaly with cold air surrounding it. Thus, the potential temperature structure will look as below.

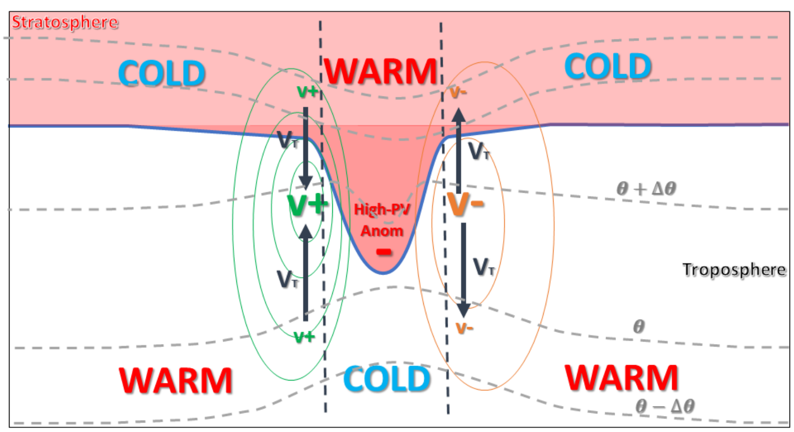


Figure 10: Cross-sectional view of the thermal structure of an idealised PV intrusion in the upper troposphere inducing upper-level cyclonic flow around it.

Low-level and surface PV anomalies

PV anomalies are not confined to the upper troposphere. High-PV structures can also be found in the low levels, often associated with diabatic processes. The PV anomalies act in a similar way to their upper-level counterparts, stimulating cyclonic flow around them. Similar arguments with respect to the thermal balance of the atmosphere can be made in order to understand the thermal structure surrounding the anomaly as well as the cyclonic flow that is induced throughout the atmosphere.

You will recall that for PV to be conserved, the flow must be both frictionless and adiabatic. At the surface, this is not strictly true. Thus, PV cannot be directly used on the surface and we need to use a PV-like parameter to analyse the surface is in a PV-thinking framework. It has been shown that surface potential temperature (θ) anomalies can act as PV-like anomalies. Warm θ anomalies behave in a similar way to high-PV anomalies on the surface where cyclonic flow is stimulated around a warm θ anomaly and anti-cyclonic flow results from cold θ anomalies.

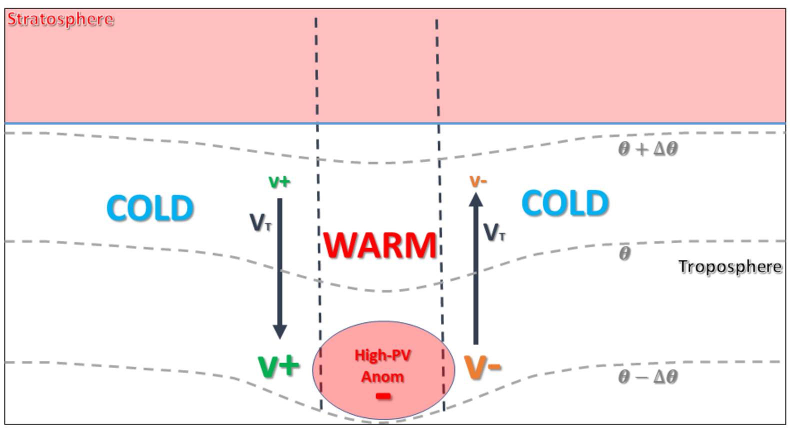


Figure 11: A cross-sectional view of an idealised PV intrusion in the lower troposphere inducing low-level cyclonic flow around it.

Interaction of anomalies

As can be shown schematically in Figure 9 and Figure 10, cyclonic circulation around the upper-level intrusion of high-PV stratospheric air into the upper troposphere is not confined to the upper troposphere. Mirrored, although weaker, cyclogenetic forcing is also present on the surface. As a result of the surface temperature gradient, the low-level cyclogenetic forcing results in warm air temperature advection ahead of the upper-level PV intrusion axis. This results in a warm potential temperature anomaly ahead of the upper-level PV axis. Recall that warm potential temperature anomalies on the surface can be interpreted to be similar to high-PV anomalies whereby they can induce cyclonic circulation around them. The cyclogenetic forcing is induced throughout the troposphere above the anomaly, with mirrored cyclogenetic forcing stimulated ahead of the upper level PV intrusion in the upper-levels. Whilst the surface anomaly lies ahead of the upper-level anomaly there is positive feedback between the two anomalies and thus are mutually beneficial to one another. Low-level anomalies induced by diabatic processes can further add to the development of the surface cyclone. The phase-locked alignment of all 3 of these anomalies is known as a "PV tower" and can lead to explosive cyclogenesis.

In the atmosphere, these processes lead to the development of baroclinic weather systems such as mid-latitude cyclones or cut-off lows that extend to the surface, where the system leans westward with height.

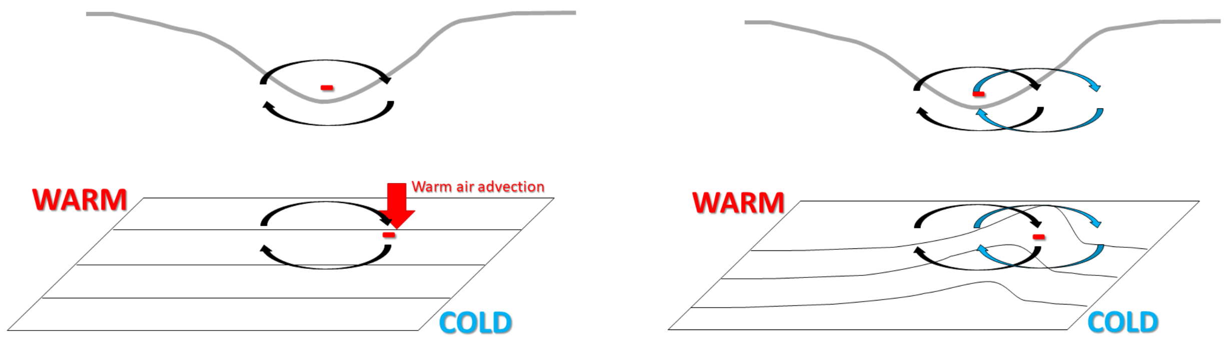


Figure 12: Adapted from Hoskins *et al.* (1985). Interaction between an upper air intrusion of high-PV air and an induced surface high-PV anomaly.

PV on isentropic maps

Isentropic surfaces, lines of constant potential temperature, are frequently used in the dynamical meteorological analyses. The analysis of isentropic PV has many applications including the identification of Rossby wave breaking (RWB) in the upper troposphere. Upper-level PV intrusions are easily identifiable on isentropic surfaces. Potential temperature contours slant surface-ward from the poles to the equator. Thus, an isentropic contour will cut through the quasi-horizontal dynamical tropopause at some point between the pole and the equator. High-PV values (stratospheric air) will be found towards the poles whilst low-PV values (tropospheric air) will be found towards the equator. A PV anomaly in the upper troposphere can be seen in the isentropic PV field as a tongue of high-PV, stratospheric air extending towards the equator.

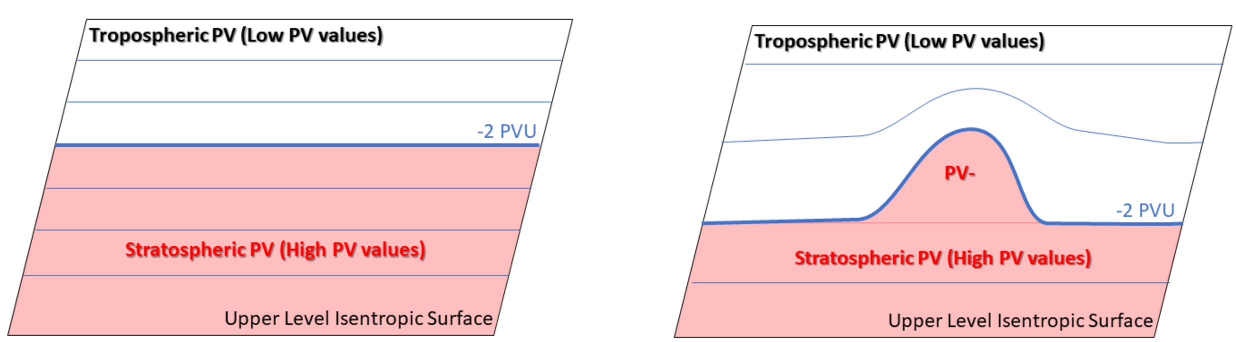


Figure 13: Left: Climatological mean isentropic surface of the upper troposphere. Right: An idealised PV intrusion in the upper troposphere as seen on upper-level isentropic PV surface.

Useful additional reading:

Lackmann (2011) - Midlatitude synoptic meteorology: Dynamics, analysis and forecasting (Chapter 4)

Hoskins *et al.* (1985) - On the use and significance of isentropic potential vorticity maps

Barnes *et al.* (2021) - Cape storm: A dynamical study of a cut-off low and its impact on South Africa

For more information on PV, follow this link: `http://www.zamg.ac.at/docu/Manual/SatManu/main.htm?/docu/Manual/SatManu/Basic/Parameters/PV.htm`

For real world examples related to the material above, follow this link: `https://weathermanbarnes.github.io/UPDynamicalForecasts`

The vorticity equation

Objective: Derive an equation for the time rate of change of vorticity without limiting the validity to adiabatic motion.

Cartesian coordinate form

Approximate horizontal momentum equations:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{[zonal component equation]}$$
$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{[meridional component equation]}$$

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) = \frac{\partial}{\partial y}(fv) - \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$
$$\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = -\frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial y}(fv) = -\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial p}{\partial x}\right)$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial x}(fu) = -\frac{\partial}{\partial x}\left(\rho^{-1}\frac{\partial p}{\partial y}\right)$$

$$\therefore \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y \partial z} - v\frac{\partial f}{\partial y} - f\frac{\partial v}{\partial y}$$
$$= -\frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} \qquad (1)$$

and

$$\therefore \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial x \partial z} + u\overbrace{\frac{\partial f}{\partial x}}^{=0} + f\frac{\partial u}{\partial x}$$
$$= -\frac{\partial \rho^{-1}}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} \qquad (2)$$

(2) − (1): LHS

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right]+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+v\frac{\partial f}{\partial y}$$

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial x}\qquad y=\rho^{-1}\quad u=\rho$$
$$\frac{\partial}{\partial x}\rho^{-1}=\frac{\partial}{\partial \rho}\rho^{-1}\frac{\partial \rho}{\partial x}$$
$$=-\rho^{-2}\frac{\partial \rho}{\partial x}$$

(2) − (1): RHS

$$-\frac{\partial \rho^{-1}}{\partial x}\frac{\partial p}{\partial y}-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}=-(-1)\rho^{-2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}+(-1)\rho^{-2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$
$$=\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$

Consider

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Since $\zeta=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$
$$+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Since $f=f(y)$, $\frac{Df}{Dt}=0+0+v\frac{\partial f}{\partial y}+0$

$$\therefore \frac{D\zeta}{Dt}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{Df}{Dt}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\implies \frac{D}{Dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)\qquad(4.17)$$

The rate of change of absolute vorticity following the motion is given by the sum of the divergence, the tilting or twisting, and the solenoidal terms.

Scale analysis of the vorticity equation

Characteristic scales for the field variables based on typical observed magnitudes for **synoptic-scale** motions:

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical scale
$L \sim 10^6\,\mathrm{m}$	length scale
$H \sim 10^4\,\mathrm{m}$	depth scale
$\delta p \sim 10\,\mathrm{hPa}$	horizontal pressure scale
$\rho \sim 1\,\mathrm{kg\,m^{-3}}$	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
$L/U \sim 10^5\,\mathrm{s}$	time scale
$f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$	Coriolis parameter
$\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$	"beta" parameter

Use an advective time scale because the vorticity pattern tends to move at a speed comparable to the horizontal wind speed.

First, the relative vorticity equation $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \lesssim \dfrac{U}{L} \sim (10^5\,\mathrm{s})^{-1} = 10^{-5}\,\mathrm{s}^{-1}$

[$\lesssim$ means less than or equal to in order of magnitude]

The magnitude of the terms of the equation below will be evaluated:

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} + w\frac{\partial\zeta}{\partial z} + (\zeta+f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

The Rossby number $\mathrm{Ro} \equiv \dfrac{U}{f_0 L}$

and since $\zeta \lesssim \dfrac{U}{L}$, $\dfrac{\zeta}{f_0} \lesssim \dfrac{U}{f_0 L} \sim 10^{-1}$ $\left[10\,\mathrm{m\cdot s^{-1}}/(10^4\,\mathrm{s^{-1}}\,10^6\,\mathrm{m})\right]$

$$\therefore \zeta \sim \frac{1}{10} f_0$$

$$\therefore (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Note: Small Ro signifies a system which is strongly affected by Coriolis forces, and a large Ro a system in which inertial and centrifugal forces dominate. In tornadoes $\mathrm{Ro} \approx 10^3$; in low pressure systems $\mathrm{Ro} \approx$ 0.1–1.

Near the centre of intense cyclonic storm $\left|\dfrac{\zeta}{f}\right| \sim 1$, the relative vorticity should be retained.

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} &\sim \frac{U/L}{L/U} = \frac{U^2}{L^2} \sim 10^{-10}\,\mathrm{s^{-2}} \qquad \left[\frac{(10\,\mathrm{m\,s^{-1}})^2}{(10^6\,\mathrm{m})^2} = 10^{-10}\,\mathrm{s^{-2}}\right] \\
u\frac{\partial \zeta}{\partial x} &\sim U\frac{U}{L}\frac{1}{L} = \frac{U^2}{L^2} \\
v\frac{\partial \zeta}{\partial y} &\sim U\frac{U}{L}\frac{1}{L} = \frac{U^2}{L^2} \\
w\frac{\partial \zeta}{\partial z} &\sim W\frac{U}{L}\frac{1}{H} \sim \frac{10^{-2}\,\mathrm{m\,s^{-1}}\,10\,\mathrm{m\,s^{-1}}}{10^6\,\mathrm{m}\,10^4\,\mathrm{m}} = 10^{-11}\,\mathrm{s^{-2}} \\
v\frac{df}{dy} &\sim U\beta \sim 10\,\mathrm{m\,s^{-1}}\,10^{-11}\,\mathrm{m^{-1}\,s^{-1}} = 10^{-10}\,\mathrm{s^{-2}}
\end{aligned}
$$

$$
\begin{cases}
f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim f_0\left(\frac{U}{L} + \frac{U}{L}\right) \sim f_0\frac{U}{L} \sim 10^{-4}\,\mathrm{s^{-1}}\,10\,\mathrm{m\,s^{-1}}\,10^{-6}\,\mathrm{m^{-1}} = 10^{-9}\,\mathrm{s^{-2}} \\
\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \lesssim \frac{W}{L}\frac{U}{H} \sim \frac{10^{-2}\,\mathrm{m\,s^{-1}}\,10\,\mathrm{m\,s^{-1}}}{10^6\,\mathrm{m}\,10^4\,\mathrm{m}} = 10^{-11}\,\mathrm{s^{-2}} \\
\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \lesssim \frac{1}{\rho^2}\left(\frac{\delta\rho}{L}\frac{\delta p}{L}\right) = \frac{\delta\rho}{\rho^2}\frac{\delta p}{L^2} \sim \frac{10^{-2}}{\rho}\frac{10\,\mathrm{hPa}}{10^{12}\,\mathrm{m^2}} \sim \frac{10^{-2}10^3\,\mathrm{Pa}}{1\,\mathrm{kg\,m^{-3}}\,10^{12}\,\mathrm{m^2}}
\end{cases} \qquad (*)
$$

Consider

$$
\begin{aligned}
1\,\mathrm{Pa} &= 1\,\mathrm{N\,m^{-2}} \\
&= 1(\mathrm{kg\,m\,s^{-2}})\,\mathrm{m^{-2}}
\end{aligned}
$$

$$
\therefore \frac{\delta\rho}{\rho^2}\frac{\delta p}{L^2} \sim 10^{-11}\frac{\mathrm{kg\,m^{-1}\,s^{-2}}}{\mathrm{kg\,m^{-1}}} = 10^{-11}\,\mathrm{s^{-2}}
$$

(*): The inequality ($\lesssim$) is used here because in each case it is possible that the two parts of the expression **might partially cancel** so that the actual magnitude would be less than indicated.

If $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are not nearly equal and opposite (i.e., divergence > 0) the divergence term would be an order of magnitude greater than the other terms (because $f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim 10^{-9}\,\mathrm{s^{-2}}$, followed by terms $\lesssim 10^{-10}$ and smaller).

Therefore, scale analysis of the vorticity equation indicates that synoptic-scale motions must be quasi-nondivergent. The divergence term will be small enough to be balanced by the vorticity advection terms only if: $\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \lesssim 10^{-6}\,\mathrm{s^{-1}}$ since $f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$, $f_0\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \sim 10^{-10}\,\mathrm{s^{-2}}$

$\implies$ The horizontal divergence must be small compared to the vorticity in synoptic-scale systems.

Retaining only the terms of order $10^{-10}\,\mathrm{s^{-2}}$ in the vorticity equation:

$$
\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = 0
$$

Remember that $v\dfrac{df}{dy} = \dfrac{Df}{Dt}$ and for horizontal motion $v\dfrac{df}{dy} = \dfrac{D_h f}{Dt}$

$$\therefore \frac{D_h \zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{D_h f}{Dt} = 0 \qquad \left[\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]$$

$$\Longrightarrow \frac{D_h}{Dt}(\zeta + f) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.22a}$$

for synoptic-scale motions.

In intense cyclonic storms $|\zeta/f| \sim 1$:

$$\Longrightarrow \frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.22b}$$

Equation (4.22a) states that the change of absolute vorticity following the horizontal motion on the synoptic scale is given approximately by the concentration or dilution of planetary vorticity caused by the convergence or divergence of the horizontal flow, respectively. In (4.22b), however, it is the concentration or dilution of absolute vorticity that leads to changes in absolute vorticity following the motion.

The form of the vorticity equation given in (4.22b) also indicates why cyclonic disturbances can be much more intense than anti-cyclones. For a fixed amplitude of convergence, relative vorticity will increase, and the factor $(\zeta + f)$ becomes larger, which leads to even higher rates of increase in the relative vorticity. For a fixed rate of divergence, however, relative vorticity will decrease, but when $\zeta \to -f$, the divergence term on the right approaches zero and the relative vorticity cannot become more negative no matter how strong the divergence (This difference in the potential intensity of cyclones and anti-cyclones was discussed in Section 3.2.5 of Holton 4 in connection with the gradient wind approximation).

The approximate forms given in (4.22a) and (4.22b) do not remain valid, however, in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only $\sim 100\,\mathrm{km}$ and the vertical velocity scale is $\sim 10\,\mathrm{cm\,s^{-1}}$. For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

Vorticity in barotropic fluids

The barotropic (Rossby) potential vorticity equation

The velocity divergence form of the continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}\cdot\vec{V} = 0 \tag{2.31}$$

For a homogenous incompressible fluid $\dfrac{D\rho}{Dt} = 0$

$$\begin{aligned}
\therefore \vec{\nabla}\cdot\vec{V} &= 0\\
\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\\
\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\frac{\partial w}{\partial z}\\
\therefore \frac{D_h}{Dt}(\zeta + f) &= -(\zeta + f)\left(-\frac{\partial w}{\partial z}\right)
\end{aligned}$$

In a barotropic fluid, we let the vorticity be approximated by ζ_g and the wind by (u_g, v_g).

$$\frac{D_h}{Dt}(\zeta_g + f) = (\zeta_g + f)\frac{\partial w}{\partial z}$$

Integrate vertically from z_1 to z_2:

$$\begin{aligned}
\int_{z_1}^{z_2}\frac{D_h}{Dt}(\zeta_g + f)dz &= \int_{z_1}^{z_2}(\zeta_g + f)\frac{\partial w}{\partial z}dz\\
h\frac{D_h}{Dt}(\zeta_g + f) &= (\zeta_g + f)\left[w(z_2) - w(z_1)\right]\\
\text{where}\quad h = h(x, y, t); &\qquad w \equiv \frac{Dz}{Dt}\\
= z_2 - z_1 &
\end{aligned}$$

$$\therefore w(z_2) - w(z_1) = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{D_h h}{Dt}$$

Since $h\frac{D_h}{Dt}(\zeta_g + f) = (\zeta_g + f)\left[w(z_2) - w(z_1)\right] = (\zeta_g + f)\frac{D_h h}{Dt}$

$$\therefore \frac{1}{(\zeta_g + f)}\frac{D_h}{Dt}(\zeta_g + f) = \frac{1}{h}\frac{D_h h}{Dt}$$
$$\therefore \frac{D_h}{Dt}(\ln(\zeta_g + f)) - \frac{D_h}{Dt}(\ln h) = 0$$
$$\therefore \frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0 \tag{4.26}$$

which is the potential vorticity conservation theorem for a barotropic fluid.

The quantity conserved following the motion in (4.26) is the Rossby potential vorticity.

Note the following:

$$\frac{D}{Dt}(\ln(\zeta_g + f)) - \frac{D}{Dt}(\ln h) = 0$$

Therefore,

$$\frac{D}{Dt}(\ln(\zeta_g + f) - \ln h) = 0$$
$$\therefore \frac{D}{Dt}\left(\ln\frac{\zeta_g + f}{h}\right) = 0$$

From calculus: $D_x \ln[f(x)] = \dfrac{f'(x)}{f(x)}$

$$\implies \frac{D}{Dt}\left(\ln\frac{\zeta_g + f}{h}\right) = \frac{1}{\frac{\zeta_g + f}{h}}\frac{D}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$
$$\therefore \frac{D}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

Exercise: By considering the essence of potential vorticity (a measure of the constant ratio of the absolute vorticity to the effective depth of the vortex), an air column at $60°$S with initial relative vorticity equal to zero, stretches from sea-level to a fixed tropopause level of $10\,\mathrm{km}$ in height. If the air column moves until it is over a mountain range $2.5\,\mathrm{km}$ high at $45°$S, what is its 1) absolute vorticity and 2) relative vorticity as it passes the mountain top?

Solution: $\dfrac{\zeta + f}{H} = \text{constant} \implies \left(\dfrac{\zeta + f}{H}\right)_{\text{initial}} = \left(\dfrac{\zeta + f}{H}\right)_{\text{final}}$, and $\zeta_{\text{initial}} = 0$ (given)

$$f_{\text{initial}} = 2\Omega\sin(-60°) = -1.263 \times 10^{-4}\,\mathrm{s}^{-1}$$
$$f_{\text{final}} = 2\Omega\sin(-45°) = -1.031 \times 10^{-4}\,\mathrm{s}^{-1}$$

$$\therefore (\zeta + f)_{\text{final}} = \frac{H_{\text{final}}}{H_{\text{initial}}} f_{\text{initial}}$$
$$= \frac{10 - 2.5}{10}(-1.263 \times 10^{-4})$$
$$= -9.473 \times 10^{-5}\,\mathrm{s}^{-1}$$

$$\therefore \zeta_{\text{final}} = -9.473 \times 10^{-5}\,\text{s}^{-1} - (-1.031 \times 10^{-4}\,\text{s}^{-1})$$
$$= 8.37 \times 10^{-6}\,\text{s}^{-1}$$

The barotropic vorticity equation

(4.23): $\frac{D_h}{Dt}(\zeta + f) = (\zeta + f)\frac{\partial w}{\partial z}$

If the flow is purely horizontal, as is the case for barotropic flow in a fluid of constant depth, the divergence term vanishes since $w = 0$.

As before, let vorticity be approximated by ζ_g:

$$\frac{D_h}{Dt}(\zeta_g + f) = 0$$

which states that absolute vorticity is conserved following the horizontal motion.

More generally, absolute vorticity is conserved for any fluid layer in which the divergence of the horizontal wind vanishes, without the requirement that the flow be geostrophic.

For horizontal motion that is nondivergent the flow can be represented by a stream function $\psi(x, y)$ such that $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$.

$$\begin{aligned}
\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) \\
&= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \\
&\equiv \nabla^2 \psi
\end{aligned}$$

Not a requirement for flow to be geostrophic: $\frac{D_h}{Dt}(\zeta + f) = 0$

$$\therefore \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$
$$\therefore \frac{\partial}{\partial t}\nabla^2\psi + \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + \vec{V}_\psi \cdot \vec{\nabla}(f) = 0; \qquad \vec{V}_\psi = \vec{k} \times \vec{\nabla}\psi$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi + f) \tag{4.28}$$

$\Longrightarrow$ The local tendency of relative vorticity is given by the advection of absolute vorticity.

Because the flow in the mid-troposphere is often nearly nondivergent on the synoptic scale, (4.28) provides a good model for short-term forecasts of the synoptic-scale $500\,\text{hPa}$ flow field.

Bjerknes-Holmboe theory

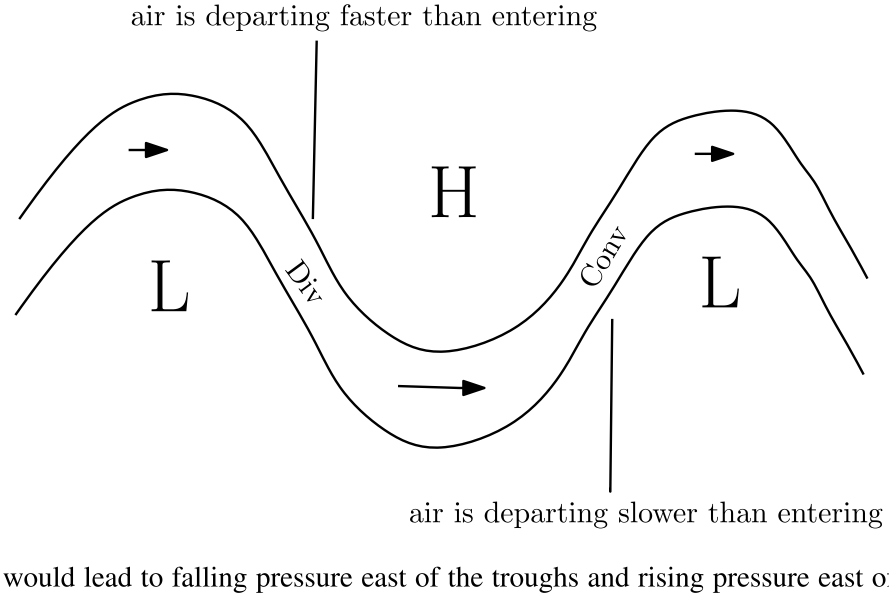
As an introduction to the notion of mid-latitude developing baroclinic systems, we introduce a theory of relating the horizontal distribution of divergence and convergence to a pattern of high and low pressure systems. Highs will move towards regions of convergence (rising pressure), and lows towards regions of divergence (falling pressure). This theory is commonly known as the Bjerknes-Holmboe theory. Here we discuss it only from a qualitative point of view.

Since the divergence of the geostrophic wind (V_g) is zero (for constant f), and the divergence of the gradient wind (V) is not zero, we will examine the pattern of divergence of idealistic pressure fields for gradient flow. Our weather pattern has

1. Sinusoidal $500\mathrm{hPa}$ contours extending from west to east,
2. Circular concentric isobars at the surface.

The curvature effect

We have already shown that $V_g > V$ for cyclonic flow, and $V_g < V$ for anticyclonic flow. Therefore, owing to this curvature effect, we expect a distribution of wind speeds as shown in this figure (the arrows represent the gradient wind).



Such a pattern would lead to falling pressure east of the troughs and rising pressure east of the ridge. The expectation is for the pressure system to move eastward because the lows (highs) move towards regions of falling (rising) pressure. Moreover, for a given fixed amplitude of such systems, short wavelengths and high wind speeds, the curvature effect results in an eastward moving wave.

The latitude effect

Assume that all other parameters are kept constant, then the geostrophic and gradient wind speeds decrease with increasing (equatorward) latitude. To demonstrate this statement, consider the gradient wind equation

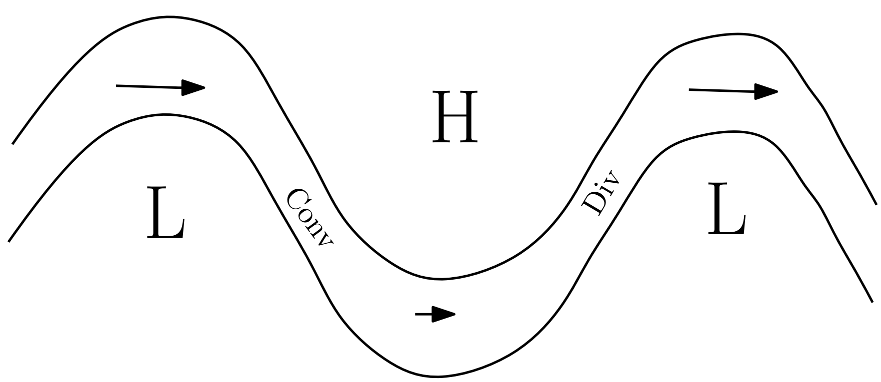
$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$
$$\therefore V = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial n} + \frac{V^2}{R}\right)$$

Apply scale analysis to $\left|\frac{V^2}{R}\right| \simeq \frac{(10\,\mathrm{m\,s^{-1}})^2}{10^6\,\mathrm{m}} = 10^{-4}\,\mathrm{m\,s^{-2}}$, the centrifugal force.

Since a typical parameter value for $\left|\frac{\partial \Phi}{\partial n}\right| = 10^{-3}\,\mathrm{m\,s^{-2}}$, the centrifugal force is about a tenth of the pressure gradient force. This result implies that $V \simeq -\frac{1}{f}\frac{\partial \Phi}{\partial n} = V_g$, which further implies that both the gradient and geostrophic wind increase or decrease similarly for a variable Coriolis parameter. Consider the following table of approximate gradient wind speeds with increasing latitude in the Southern Hemisphere

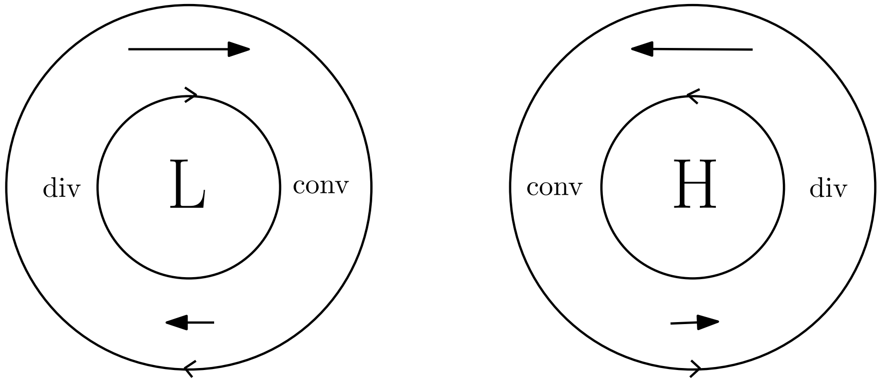
Latitude	Approximate gradient wind speed $(\mathrm{m\,s^{-1}})$
-30	13.7
-45	9.7
-60	7.9

From the gradient wind relationship and the table above, wind speed decreases with increasing latitude.



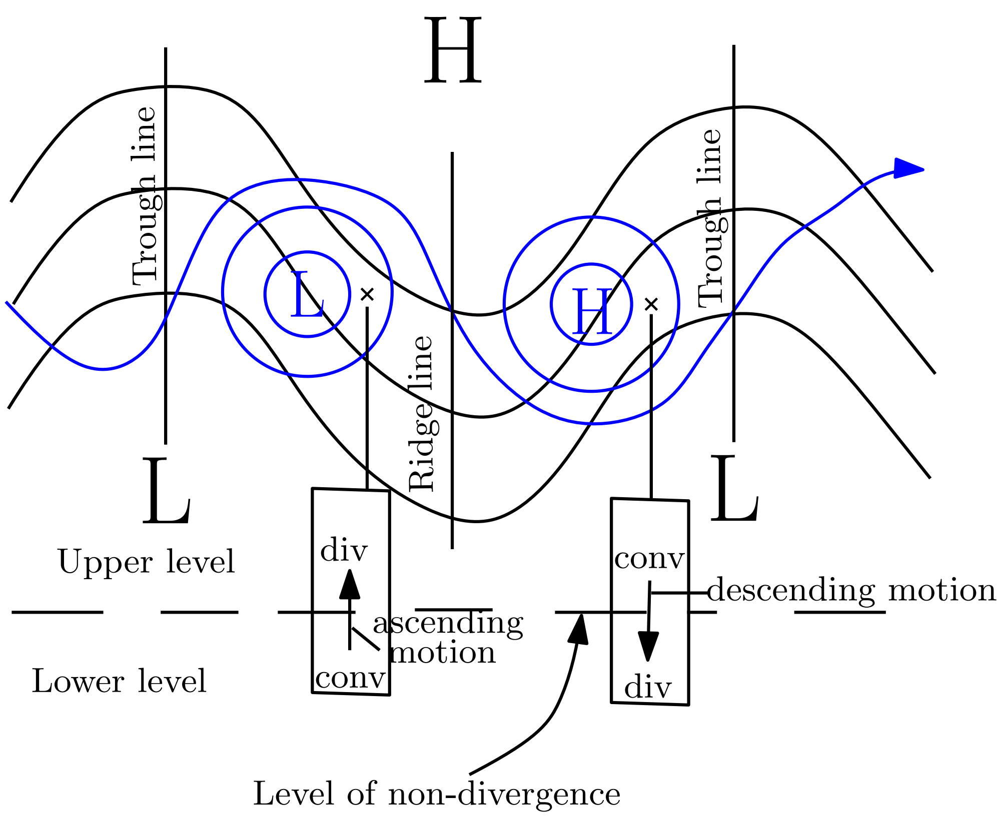
For low wind speeds and long wavelengths, the curvature term may be small, resulting in the wave moving westward as determined by the latitude effect. This effect is enforced when the wave amplitude is large.

Next, consider the case of equally spaced, concentric, circular isobars of a surface low and high pressure system in the Southern Hemisphere, which results in the curvature effect to be the same everywhere. However, the latitude effect will produce higher winds on the equatorward side.



The result of the latitude effect is convergence with rising pressure to the east (west) of the low (high) pressure and divergence with falling pressure to the west (east). Such systems are expected to move westward.

The idealized model



For the usual short-wave systems in which the curvature effect dominates the latitude effect, divergence is found east of the trough line and convergence ahead of the ridge line. East of the centre of the surface low, low-level convergence is found with divergence aloft, resulting in ascending motion. The opposite is found east of the ridge line where upper-level convergence is associated with low-level divergence east of the surface high, resulting in descending motion.

Consider the level of non-divergence shown in the figure. This is a level of transition from the positive to negative divergence, and vice versa. If this level is low in altitude, the high altitude pattern will predominate, and the system will move eastward. If this level is high in altitude, the low altitude pattern will predominate, and the system will move westward.

We have introduced here a classic theory qualitatively of the motion of pressure systems in mid-latitudes. Although this theory may reveal considerable quantitative agreement with synoptic experience, also over the Southern Hemisphere, we will develop and discuss quantitatively a set of equations that are less complicated than the full set of primitive equations of motion in order to describe extra-tropical weather systems. This set of equations represent the so-called quasi-geostrophic approximation. Why the theory is called quasi-geostrophic? It is because if the winds in mid-latitude systems were perfectly geostrophic, such winds never cross the isobars, and could thus not cause convergence into the low pressure system and therefore no vertical velocity. Since we know from observations that vertical motion does exist and are important for causing clouds and rain development in cyclones, the upward motion cannot be geostrophic. By including this ageostrophic flow into the set of equations that are otherwise totally geostrophic, the equations are said to be quasi-geostrophic, meaning partially geostrophic.

The quasi-geostrophic approximation

To show that for motions that are hydrostatic and nearly geostrophic, the 3-dimensional flow field is determined approximately by the **isobaric** distribution of geopotential $[\Phi(x, y, p, t)]$.

The use of the isobaric coordinate system simplifies the development of approximate prognostic and diagnostic equations.

Scale analysis in isobaric coordinates

Horizontal momentum equation	$\dfrac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} = -\vec{\nabla}\Phi$	(3.2) also (6.1)
Hydrostatic equation	$\dfrac{\partial \Phi}{\partial p} = -\alpha = -\dfrac{RT}{p}$	(3.27) also (6.2)
Continuity equation	$\vec{\nabla} \cdot \vec{V} + \dfrac{\partial \omega}{\partial p} = 0$	(3.5) also (6.3)
Thermodynamic energy equation	$\left(\dfrac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}\right) T - S_p \omega = \dfrac{J}{c_p}$	(3.6) also (6.4)

Total derivative in (3.2):

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + \left(\vec{V} \cdot \vec{\nabla}\right)_p + \omega \frac{\partial}{\partial p} \qquad \left[\omega \equiv \frac{Dp}{Dt}\right] \tag{6.5}$$

From (3.6): $S_p \equiv -T \dfrac{\partial \ln \theta}{\partial p}$, static stability parameter $\left[S_p \sim 5 \times 10^{-4}\,\mathrm{K\,Pa^{-1}}\ \text{ in mid-troposphere}\right]$

The above set of equations still contain several terms that are of secondary significance for mid-latitude synoptic-scale systems. They can be simplified further by

1) horizontal flow is nearly geostrophic

2) the magnitude of the ratio of vertical velocity to horizontal velocity is of the order 10^{-3}.

Separate the horizontal velocity into geostrophic and ageostrophic parts:

$$\vec{V} = \vec{V}_g + \vec{V}_a;\ \vec{V}_g \equiv \frac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi \qquad \left[\vec{V}_a = \vec{V} - \vec{V}_g\right] \tag{6.7}$$

Regarding f_0: It is assumed that the meridional length scale (L) is small compared to the radius of the Earth so that the geostrophic wind (6.7) may be defined using a constant reference latitude value of the Coriolis parameter.

For the systems of interest

$$\left|\vec{V}_g\right| \gg \left|\vec{V}_a\right| \quad \text{or} \quad \frac{\left|\vec{V}_a\right|}{\left|\vec{V}_g\right|} \sim O(Ro), \quad \text{that is the same order of magnitude as the Rossby number}$$

$$\left(Ro \equiv \frac{U}{f_0 L} \sim 0.1 \text{ from Page 41 of Holton 4}\right)$$

Momentum can then be approximated to $O(Ro)$ by its geostrophic value, and the rate of change of momentum (or temperature) following the horizontal motion can be approximated to the same order by the rate of change following the geostrophic wind.

In equation (6.5):

1) $\vec{V}$ can be replaced by $\vec{V}_g$
2) the vertical advection which arises only from the ageostrophic flow can be neglected.

$$\therefore \frac{D\vec{V}}{Dt} \approx \frac{D_g\vec{V}_g}{Dt}$$

where

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y} \tag{6.8}$$

Note: Newton's second law; a form of the momentum equation:

$$\frac{D\vec{U}}{Dt} = -2\vec{\Omega} \times \vec{U} - \frac{\vec{1}}{\rho}\vec{\nabla}p + \vec{g} + \vec{F}_r \tag{2.8}$$

The dynamical effect of the variation of the Coriolis parameter with latitude needs to be retained in the Coriolis force term in the momentum equation. This variation can be approximated using a Taylor series:

$$f = f_0 + \left(\frac{df}{dy}\right)_{\phi_0} (y + y_0) + \text{higher order terms}$$

$$\beta \equiv \left(\frac{df}{dy}\right)_{\phi_0}, y = 0 \text{ at } \phi_0$$

This approximation is referred to as the: **mid-latitude β–plane** approximation

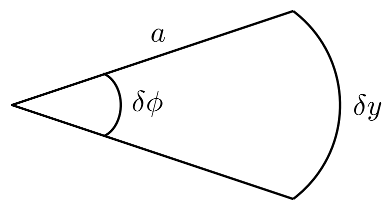
$$f = f_0 + \beta y \tag{6.9}$$

f_0 is the Coriolis parameter computed at a characteristic latitude, ϕ_0; the variable y measures the meridional distance from this latitude.

$$\begin{aligned}\beta &= \left(\frac{df}{dy}\right)_{\phi_0} \\ &= \frac{d}{dy}(2\Omega \sin\phi)_{\phi_0}\end{aligned}$$

From the figure below:

$$\delta y = a\delta\phi$$
$$\therefore \frac{1}{\delta y} = \frac{1}{a}\frac{1}{\delta\phi}$$



$$\therefore \beta = \frac{1}{a}\frac{d}{d\phi}(2\Omega\sin\phi)_{\phi_0}$$
$$= \frac{2\Omega\cos\phi_0}{a}$$

The ratio of the terms on the right of (6.9):

$$\frac{\beta y}{f_0} \sim \frac{\beta L}{f_0} \sim \frac{2\Omega\cos\phi_0}{a}L\frac{1}{2\Omega\sin\phi_0}$$
$$= \frac{\cos\phi_0 L}{\sin\phi_0 a} \sim O(Ro) \ll 1 \quad \left[\text{Note: } \frac{\cos(-45°)}{\sin(-45°)} = -1 \text{ and } L \text{ is small compared to the radius of the Earth, } a\right]$$

$\therefore f_0 \gg \beta y$, which justifies letting the Coriolis parameter be a constant f_0 in the geostrophic approximation and using (6.9)

(6.1): $\frac{D\vec{V}}{Dt} + f\vec{k}\times\vec{V} + \vec{\nabla}\Phi = 0$ (the acceleration following the motion, the Coriolis force and the pressure gradient force are balanced)

Consider

$$f\vec{k}\times\vec{V} + \vec{\nabla}\Phi = (f_0 + \beta y)\vec{k}\times(\vec{V}_g + \vec{V}_a) + \vec{\nabla}\Phi$$
$$= f_0\vec{k}\times\vec{V}_g + \beta y\vec{k}\times\vec{V}_g + f_0\vec{k}\times\vec{V}_a + \beta y\vec{k}\times\vec{V}_a - f_0\vec{k}\times\vec{V}_g$$
$$= f_0\vec{k}\times\vec{V}_a + \beta y\vec{k}\times\vec{V}_g + \beta y\vec{k}\times\vec{V}_a$$

Neglect the ageostrophic wind compared to the geostrophic wind in the term proportional to βy:

Atmospheric waves influenced by the beta (β) term are characterized as planetary waves (also called Rossby waves). These waves experience the curvature of a revolving planet through meridional changes in the Coriolis parameter. The so-called beta effect may be considered to be small when a synoptic-scale storm moves across only a small range of latitudes during its lifetime.

$$\therefore f\vec{k}\times\vec{V} + \vec{\nabla}\Phi \approx f_0\vec{k}\times\vec{V}_a + \beta y\vec{k}\times\vec{V}_g \tag{6.10}$$

The horizontal momentum equation i.t.o. geostrophic flow then becomes:

$$\frac{D_g\vec{V}_g}{Dt} = -f_0\vec{k}\times\vec{V}_a - \beta y\vec{k}\times\vec{V}_g$$

and each of these terms is $O(Ro)$ compared to the pressure gradient force, and the neglected terms are $O(Ro^2)$ or smaller.

$f_o \gg \beta y$ and $|\overline{V}_g| \gg |\overline{V}_a|$

$\therefore \beta y \sim \frac{1}{10} f_o$ $\quad\therefore |\overline{V}_a| \sim \frac{1}{10}|\overline{V}_g|$

$$\begin{aligned}\frac{D_g\overline{V}_g}{Dt} &= -f_o\overline{k}x\overline{V_a} - \beta y\overline{k}x\overline{V_g}\\ &\sim - f_o\overline{k}x\left(\frac{1}{10}\overline{V_g}\right) - \frac{1}{10}f_o\overline{k}x\overline{V_g}\\ &\sim -\frac{1}{10}(f_o\overline{k}\,x\overline{V}_g)\\ &= -\frac{1}{10}(-\overline{\nabla}\Phi)\\ &= \frac{1}{10}\overline{\nabla}\Phi\end{aligned}$$

(6.7): $f_o\overline{V}_g = \overline{k}\,x\overline{\nabla}\Phi$

$\therefore \overline{\nabla}\Phi = -f_o\overline{k}\,x\overline{V}_g$

since $\overline{k}\,x(\overline{k}\,x\overline{\nabla}\Phi)$

$= -\overline{\nabla}\Phi$

Next, $\vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot(\vec{V}_g+\vec{V}_a) = \vec{\nabla}\cdot\vec{V}_g + \vec{\nabla}\cdot\vec{V}_a$

Since $\vec{V}_g = \frac{1}{f_0}\vec{k}\times\vec{\nabla}\Phi$ is non-divergent, $\vec{\nabla}\cdot\vec{V}_g = 0$

$$\therefore \vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot\vec{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

$$(6.3) : \vec{\nabla}\cdot\vec{V} + \frac{\partial\omega}{\partial p} = 0 \implies \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial\omega}{\partial p} = 0 \qquad (6.12)$$

(6.12) means that ω is determined only by the ageostrophic part of the wind field.

The thermodynamic energy equation (6.4): $\left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)T - S_p\omega = \frac{J}{c_p}$

However, the **horizontal advection** can be approximated by the geostrophic value

$$\therefore \left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right) T - S_p\omega = \frac{J}{c_p}$$

The **vertical advection** is not neglected and forms part of the adiabatic heating and cooling term. This term must be retained because the static stability is usually large enough on the synoptic scale so that the adiabatic heating/cooling due to vertical motion is of the same order as the horizontal temperature advection.

Simplifying the adiabatic heating and cooling term: Divide the total temperature field, T_{tot}, into a basic state (standard atmosphere) portion that depends only on pressure, $T_0(p)$, plus a deviation from the basic state, $T(x, y, p, t)$.

$$T_{tot}(x, y, p, t) = \underbrace{T_0(p)}_{\text{Basic state}} + \overbrace{T(x, y, p, t)}^{\text{Deviation from the basic state}}$$

Static stability parameter in the isobaric system

$$S_p \equiv -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \tag{3.7}$$

$$S_p = -T\frac{\partial \ln \theta}{\partial p} = -T_0\frac{\partial \ln \theta}{\partial p}$$

because $\left|\frac{dT_0}{dp}\right| \gg \left|\frac{\partial T}{\partial p}\right|$

θ_0 is the potential temperature that corresponds to the basic state temperature T_0, which is only a function of p $[T_0 = T_0(p)]$.

$$\therefore \frac{\partial \ln \theta_0}{\partial p} = \frac{d \ln \theta_0}{dp}$$

$$\begin{aligned}\therefore S_p = -T_0\frac{d \ln \theta_0}{dp} &= -T_0\frac{d \ln \theta_0}{dp}\left(\frac{p}{R}\right)\left(\frac{R}{p}\right)\\ &= -\frac{RT_0}{p}\frac{d \ln \theta_0}{dp}\left(\frac{p}{R}\right)\end{aligned}$$

$$\sigma \equiv -\frac{RT_0}{p}\frac{d \ln \theta_0}{dp}$$

$$\therefore S_p = \frac{\sigma p}{R}$$

$$\therefore \left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right) T - \left(\frac{\sigma p}{R}\right) \omega = \frac{J}{c_p}$$

(6.2): $\dfrac{\partial \Phi}{\partial p} = -\dfrac{RT}{p} \implies T = -\dfrac{p}{R}\dfrac{\partial \Phi}{\partial p}$

$$\therefore \left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right)\left(\frac{p}{R}\right)\left(\frac{\partial \Phi}{\partial p}\right) - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right)\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{R}{p}\frac{J}{c_p} = \frac{\kappa J}{p} \qquad \left[\kappa \equiv \frac{R}{c_p}\right] \tag{6.13b}$$

The quasi-geostrophic equations form a complete set in the dependent variables $\Phi, \vec{V}_g, \vec{V}_a$, and ω.

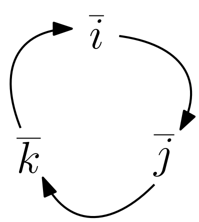
$$\vec{V}_g = \frac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi \tag{6.7}$$

$$\frac{D_g \vec{V}_g}{Dt} = -f_0 \vec{k} \times \vec{V}_a - \beta y \vec{k} \times \vec{V}_g \tag{6.11}$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{6.12}$$

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \tag{6.13b}$$

The quasi-geostrophic vorticity equation



$$\vec{V}_g = \frac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi \tag{6.7}$$

$$f_0 \vec{V}_g = f_0(u_g \vec{i} + v_g \vec{j}) = \vec{k} \times \left(\frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j}\right)$$
$$= \frac{\partial \Phi}{\partial x}\vec{j} - \frac{\partial \Phi}{\partial y}\vec{i}$$
$$\therefore f_0 v_g = \frac{\partial \Phi}{\partial x}, \quad f_0 u_g = -\frac{\partial \Phi}{\partial y} \tag{6.14}$$

Geostrophic vorticity

$$\begin{aligned}
\zeta_g = \vec{k} \cdot \vec{\nabla} \times \vec{V}_g = \vec{k} &\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u_g & v_g & 0 \end{vmatrix} \\
&= \vec{k} \cdot \left[\vec{k} \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) \right] \\
&= \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}
\end{aligned}$$

$$\begin{aligned}
\therefore \zeta_g &= \frac{\partial}{\partial x} \left(\frac{1}{f_0} \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{1}{f_0} \left(-\frac{\partial \Phi}{\partial y} \right) \right) \\
&= \frac{1}{f_0} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{1}{f_0} \nabla^2 \Phi \qquad (6.15)
\end{aligned}$$

This equation can be used to determine $\zeta_g(x, y)$ from a known field $\Phi(x, y)$.

It can also be solved by inverting the Laplacian operator to determine Φ from a known distribution of ζ_g, provided that suitable conditions on Φ are specified on the boundaries of the region in question.

Vorticity is a useful forecast diagnostic: if the evolution of the vorticity can be predicted, then inversion of (6.15) yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Note: The Laplacian of a function tends to be a maximum where the function itself is a minimum...

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi$$

It will be shown later in the course that $\nabla^2 \Phi \propto -\Phi$.

In Northern Hemisphere:

$$\begin{aligned}
\frac{1}{f_0} \nabla^2 \Phi &\propto -\Phi \quad \text{since } f_0 > 0 \\
\therefore \zeta_g &\propto -\Phi
\end{aligned}$$

$\Longrightarrow$ positive vorticity implies low values of geopotential, and vice versa.
At ridge Φ is a maximum, thus $\zeta_g < 0$
At trough Φ is a minimum, thus $\zeta_g > 0$

In Southern Hemisphere:

$$\begin{aligned}
\frac{1}{f_0} \nabla^2 \Phi &\propto \Phi \quad \text{since } f_0 < 0 \\
\therefore \zeta_g &\propto \Phi
\end{aligned}$$

$\Longrightarrow$ positive vorticity implies high values of geopotential, and vice versa.
At ridge Φ is a maximum, thus $\zeta_g > 0$
At trough Φ is a minimum, thus $\zeta_g < 0$

The quasi-geostrophic vorticity equation can be obtained from the quasi-geostrophic momentum equation (6.11):

$$\frac{D_g\vec{V}_g}{Dt} = -f_0\vec{k}\times\vec{V}_a - \beta y\vec{k}\times\vec{V}_g$$

$$\begin{aligned}\frac{D_g}{Dt}(u_g\vec{i}+v_g\vec{j}) &= -f_0\vec{k}\times(u_a\vec{i}+v_a\vec{j}) - \beta y\vec{k}\times(u_g\vec{i}+v_g\vec{j})\\ &= -f_0u_a\vec{j} - f_0(-v_a\vec{i}) - \beta yu_g\vec{j} - \beta y(-v_g\vec{i})\end{aligned}$$

$$\therefore\ \frac{D_g}{Dt}u_g = f_0v_a + \beta yv_g$$

$$\&\ \frac{D_g}{Dt}v_g = -f_0u_a - \beta yu_g$$

$$\therefore\ \frac{D_g}{Dt}u_g - f_0v_a - \beta yv_g = 0 \tag{6.16}$$

$$\&\ \frac{D_g}{Dt}v_g + f_0u_a + \beta yu_g = 0 \tag{6.17}$$

$\frac{\partial}{\partial x}(6.17) - \frac{\partial}{\partial y}(6.16)$:

$$\frac{\partial}{\partial y}\left(\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - f_0v_a - \beta yv_g\right) = 0$$

$$\frac{\partial^2u_g}{\partial y\partial t} + \frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x} + u_g\frac{\partial^2u_g}{\partial x\partial y} + \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y} + v_g\frac{\partial^2u_g}{\partial y^2} - v_a\overbrace{\frac{\partial f_0}{\partial y}}^{=0} - f_0\frac{\partial v_a}{\partial y} - \frac{\partial\beta}{\partial y}yv_g - \beta\underbrace{\frac{\partial y}{\partial y}}_{=1}v_g - \beta y\frac{\partial v_g}{\partial y} = 0 \tag{1}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} + f_0u_a + \beta yu_g\right) = 0$$

$$\frac{\partial^2v_g}{\partial x\partial t} + \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} + u_g\frac{\partial^2v_g}{\partial x^2} + \frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y} + v_g\frac{\partial^2v_g}{\partial x\partial y} + u_a\overbrace{\frac{\partial f_0}{\partial x}}^{=0} + f_0\frac{\partial u_a}{\partial x} + \frac{\partial\beta}{\partial x}yu_g + \beta\underbrace{\frac{\partial y}{\partial x}}_{=0}u_g + \beta y\frac{\partial u_g}{\partial x} = 0 \tag{2}$$

$(2) - (1)$:

$$\frac{\partial^2 v_g}{\partial x \partial t} - \frac{\partial^2 u_g}{\partial y \partial t} + u_g \frac{\partial^2 v_g}{\partial x^2} - u_g \frac{\partial^2 u_g}{\partial x \partial y} + v_g \frac{\partial^2 v_g}{\partial x \partial y} - v_g \frac{\partial^2 u_g}{\partial y^2} + f_0 \frac{\partial u_a}{\partial x} + f_0 \frac{\partial v_a}{\partial y}$$
$$+ \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y}$$
$$+ \frac{\partial \beta}{\partial x} y u_g + \frac{\partial \beta}{\partial y} y v_g + \beta v_g + \beta y \frac{\partial u_g}{\partial x} + \beta y \frac{\partial v_g}{\partial y} = 0$$
$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + u_g \frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + v_g \frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)$$
$$+ \frac{\partial u_g}{\partial x}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \frac{\partial v_g}{\partial y}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)$$
$$+ \text{“}\beta \text{ terms”} = 0$$
$$\therefore \frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)$$
$$+ \zeta_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)$$
$$+ 0 + 0 + \beta v_g + \beta y\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = 0$$

But the divergence of the geostrophic wind vanishes:

$$\vec{\nabla} \cdot \vec{V}_g = 0$$
$$\therefore \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$
$$\implies \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)\zeta_g = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g$$
$$\therefore \frac{D_g}{Dt}\zeta_g = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \tag{6.18}$$

Take note

$$\frac{D_g f}{Dt} = \underbrace{\frac{\partial f}{\partial t}}_{=0} + u_g \underbrace{\frac{\partial f}{\partial x}}_{=0} + v_g \frac{\partial f}{\partial y} \qquad [f = f(y)]$$
$$= 0 + \vec{V}_g \cdot \vec{\nabla} f = \beta v_g$$
$$\therefore \frac{\partial \zeta_g}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\zeta_g = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \vec{V}_g \cdot \vec{\nabla} f$$

From (6.12): $\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = -\frac{\partial \omega}{\partial p}$

$$\therefore \frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \vec{\nabla}\zeta_g - f_0\left(-\frac{\partial \omega}{\partial p}\right) - \vec{V}_g \cdot \vec{\nabla} f$$
$$= -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \tag{6.19}$$

In words: The local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect).

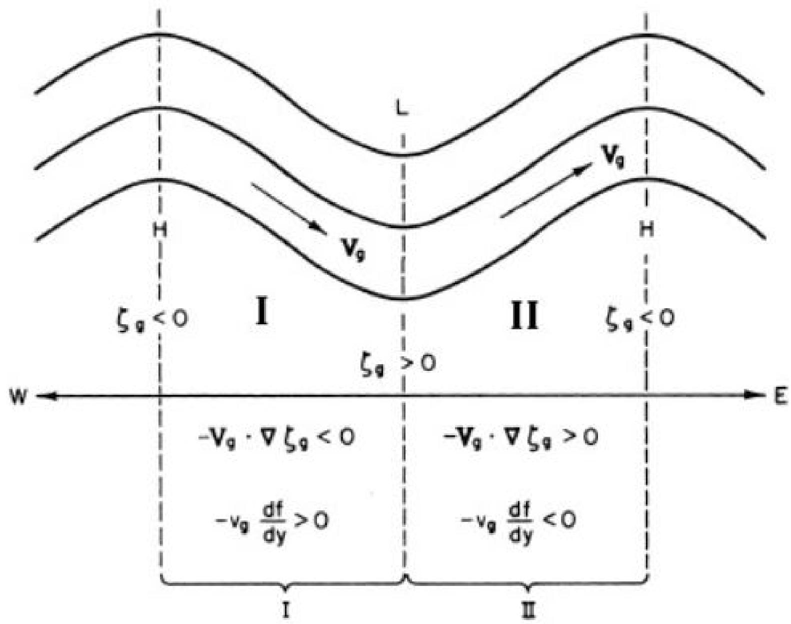
$$\text{Vorticity tendency due to vorticity advection: } -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f)$$
$$= -\vec{V}_g \cdot \vec{\nabla}\zeta_g - \beta v_g$$

$\vec{V}_g \cdot \vec{\nabla}\zeta_g$: geostrophic advection of relative vorticity

βv_g : geostrophic advection of planetary vorticity

For disturbances in the westerlies, these two effects tend to have opposite signs.

Consider the figure for an idealized 500hPa flow in the **Northern Hemisphere**.



In **region I**, upstream of the 500hPa trough, the geostrophic wind is directed from the relative vorticity minimum at the ridge towards the relative vorticity maximum at the trough.

$$\therefore \vec{V}_g \cdot \vec{\nabla}\zeta_g > 0 \implies -\vec{V}_g \cdot \vec{\nabla}\zeta_g < 0$$

At the same time $v_g < 0$ in **region I** because it is directed southwards.

Take note that $\beta = 2\Omega \cos\phi_0 / a > 0$ in both hemispheres.

$$\therefore \beta v_g < 0 \implies -\beta v_g > 0$$

We now have in region I that the:

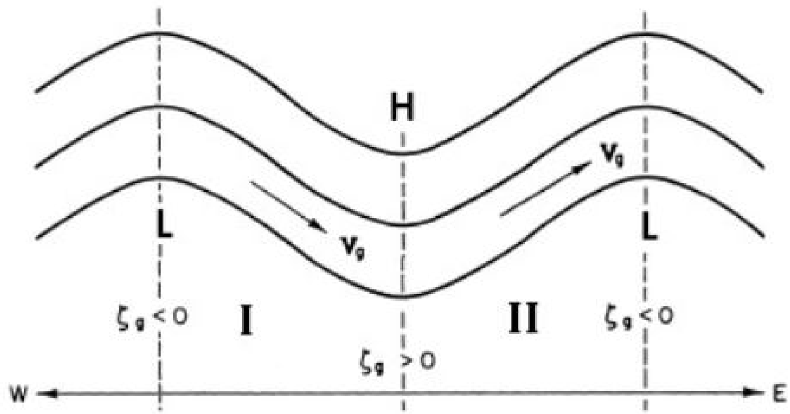
1) advection of relative vorticity tends to **decrease** the local vorticity

2) advection of planetary vorticity tends to **increase** the local vorticity.

The same arguments can be applied for region II.

Therefore, advection of relative vorticity tends to move the vorticity pattern and hence the troughs and ridges eastward (downstream). However, advection of planetary vorticity tends to move the troughs and ridges westward against the advecting wind field.

The net effect of advection on the evolution of the vorticity pattern depends on which type of vorticity advection dominates.



Consider the schematic of the 500hPa geopotential field in the Southern Hemisphere above.

The advection of the absolute vorticity by the geostrophic wind:

$$-\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) = -\vec{V}_g \cdot \vec{\nabla}\zeta_g - \beta v_g$$

Region I: Advection of relative vorticity is positive because we are going from $\zeta_g < 0$ at the trough to $\zeta_g >$ at the ridge.

$$\therefore \vec{V}_g \cdot \vec{\nabla}\zeta_g > 0 \implies -\vec{V}_g \cdot \vec{\nabla}\zeta_g < 0$$

We have shown that $\beta > 0$. However, in the region v_g points southwards. Therefore, $v_g < 0$.

$$\therefore \beta v_g < 0 \implies -\beta v_g > 0$$

Region II: $-\vec{V}_g \cdot \vec{\nabla}\zeta_g > 0$, because advection of relative vorticity is negative and $-\beta v_g < 0$ because $v_g > 0$.

Consider an idealised geopotential distribution on a mid-latitude β-plane of the form

$$\Phi(x, y) = \Phi_0 - f_0 U y + f_0 A \sin kx \cos ly$$

Φ_0, a constant zonal speed U, and amplitude A depend only on pressure. Wave numbers k and l are defined as $k = 2\pi/L_x$ and $l = 2\pi/L_y$. L_x and L_y are respectively the wavelengths in the x and y directions. y in the geopotential distribution equation is given by $a(\phi - \phi_0)$, with a the radius of the Earth and ϕ_0 the latitude at which f_0 is evaluated.

$$\begin{aligned}
\text{For } \phi - \phi_0 &= 6°,\ y = 6.67 \times 10^5\,\text{m} \\
&= 3°,\ y = 3.34 \times 10^5\,\text{m} \\
&= 1°,\ y = 1.11 \times 10^5\,\text{m} \\
&= 10°,\ y = 1.11 \times 10^6\,\text{m}
\end{aligned}$$

Therefore, for $10°$ displacement in y, y is approximately equal to the length scale, L, of 10^6 m.

$$\begin{aligned}u_g &= -\frac{1}{f_0}\frac{\partial}{\partial y}\left(\Phi_0 - f_0 U y + f_0 A \sin kx \cos ly\right)\\ &= U - A\sin kx(-l \sin ly) = U + lA\sin kx \sin ly\end{aligned}$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = \frac{1}{f_0}\left(f_0 kA\cos kx \cos ly\right) = kA\cos kx\cos ly$$

$$\zeta_g = \frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi$$

$$\frac{\partial}{\partial x}\left(f_0 kA\cos kx\cos ly\right) = -f_0 k^2 A\sin kx\cos ly$$

$$\frac{\partial}{\partial y}\left(-f_0 U - f_0 lA\sin kx\sin ly\right) = -f_0 l^2 A\sin kx\cos ly$$

$$\begin{aligned}\therefore \zeta_g &= \frac{1}{f_0}\left(-f_0k^2 - f_0 l^2\right)A\sin kx\cos ly\\ &= -(k^2+l^2)A\sin kx\cos ly\end{aligned}$$

Advection of relative vorticity:

$$\begin{aligned}-\vec{V}_g\cdot\vec{\nabla}\zeta_g &= -\left(u_g\vec{i} + v_g\vec{j}\right)\cdot\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\zeta_g\\ &= -u_g\frac{\partial\zeta_g}{\partial x} - v_g\frac{\partial\zeta_g}{\partial y}\\ &= -u_g(-(k^2+l^2)Ak\cos kx\cos ly) - v_g(-(k^2+l^2)Al\sin kx(-\sin ly))\\ &= -u_g(-(k^2+l^2)v_g) - v_g((k^2+l^2)(u_g - U))\\ &= u_g v_g(k^2+l^2) - u_g v_g(k^2+l^2) + v_g U(k^2+l^2)\\ &= v_g U(k^2+l^2) = Ak\cos kx\cos ly U(k^2+l^2)\\ &= kU(k^2+l^2)A\cos kx\cos ly\end{aligned}$$

Advection of planetary vorticity:

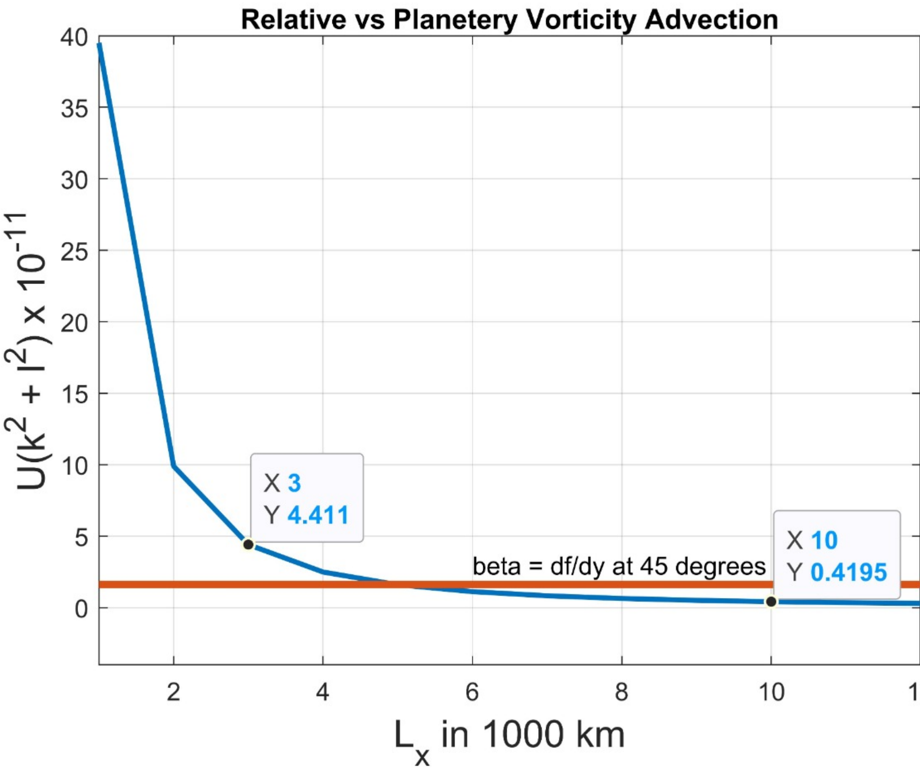
$$-v_g\frac{df}{dy} = -v_g\beta = -\beta Ak\cos kx\cos ly \quad [\beta = 2\Omega\cos\phi_0/a]$$

Advection of absolute vorticity:

$$\begin{aligned}-\vec{V}_g\cdot\vec{\nabla}(\zeta_g + f) &= kU(k^2+l^2)A\cos kx\cos ly - \beta Ak\cos kx\cos ly\\ &= kA\cos kx\cos ly(U(k^2+l^2) - \beta)\end{aligned}$$

$k = \frac{2\pi}{L_x}$, $l = \frac{2\pi}{L_y}$ with L_x and L_y the wavelengths in the x and y directions, respectively.

Consider $l = \frac{\pi}{2}\times 10^{-7}\,\mathrm{m}^{-1}$ for fixed L_y wavelengths. However, we want to determine the effect of L_x on the advection of both relative and planetary vorticity, and so wavenumber k varies with a range of L_x (i.e., $1000\,\mathrm{km}$ to $12000\,\mathrm{km}$). We therefore need to evaluate $U(k^2+l^2)$ against β as shown in the figure below.



Take note that the term representing the advection of relative vorticity ($U(k^2 + l^2)$) at $L_x = 3000\,\mathrm{km}$ is about ten times larger than the value at $L_x = 10000\,\mathrm{km}$. This result implies that relative vorticity advection is multiple times larger than planetary vorticity advection at $3000\,\mathrm{km}$ where there is a clear exponential inflection on the figure.

By considering a simplified version of an idealised geopotential distribution, a similar result is obtained.

$$\Phi(x, y, p) = \Phi_0(p) - f_0 U_0 y \sin\left(\frac{\pi p}{p_0}\right) + f_0 A \sin kx$$

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = -\frac{1}{f_0}\left(-f_0 U_0 \sin\left(\frac{\pi p}{p_0}\right)\right) = U_0 \sin\left(\frac{\pi p}{p_0}\right)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = \frac{1}{f_0}\left(f_0 A k \cos kx\right) = Ak \cos kx$$

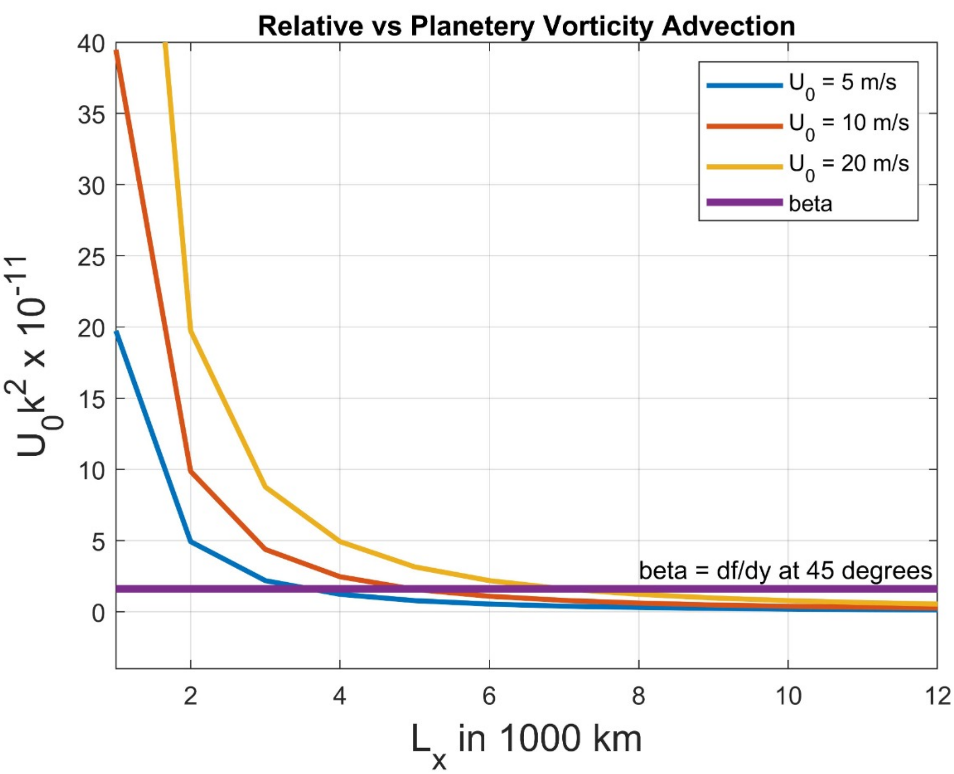
$$\zeta_g = \frac{1}{f_0}\left(\frac{\partial}{\partial x}\left(f_0 A k \cos kx\right) + \frac{\partial}{\partial y}\left(-f_0 U_0 \sin\left(\frac{\pi p}{p_0}\right)\right)\right)$$
$$= \frac{1}{f_0}\left(-f_0 A k^2 \sin kx\right) = -k^2 A \sin kx$$

$$-\vec{V}_g \cdot \vec{\nabla}\zeta_g = k^2 U_0 \sin\left(\frac{\pi p}{p_0}\right) Ak \cos kx$$

$$\therefore -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) = \left(k^2 U_0 \sin\left(\frac{\pi p}{p_0}\right) - \beta\right) Ak \cos kx$$
$$= \left(k^2 U_0 - \beta\right) Ak \cos kx$$

when $p_0 = 1000\mathrm{hPa}$ and $p = 500\mathrm{hPa}$.

Here we also show the results of having different values of U_0, the constant zonal speed. Clearly, the strength of a constant zonal wind will affect the wave lengths of short-wave systems, but will have a minimal affect on the wavelengths of Rossby waves



Exercise 1: Suppose that on the $500\mathrm{hPa}$ surface of the schematic above, the relative vorticity at a certain location at $45°$S latitude is increasing at a rate of $3 \times 10^{-6}\,\mathrm{s}^{-1}$ per 3 hours. The wind is from the northwest at $20\,\mathrm{m\,s}^{-1}$ and the relative vorticity increases towards the southeast at a rate of $4 \times 10^{-6}\,\mathrm{s}^{-1}$ per $100\,\mathrm{km}$. Use the quasi-geostrophic vorticity equation to estimate the horizontal divergence at this location on a β-plane.

Make use of the following assumptions:

1. The constant Coriolis parameter is equal to $-10^{-4}\,\mathrm{s}^{-1}$ in the Southern Hemisphere
2. β is approximated by $10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$
3. The following relationship is valid for natural coordinates: $\vec{V}_g \cdot \vec{\nabla}\zeta_g \sim v\frac{\partial \zeta_g}{\partial s}$, where s is the distance along the curve ($500\mathrm{hPa}$ contour)

Solution:

$$\begin{aligned}
\frac{\partial \zeta_g}{\partial t} &= -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0(-\vec{\nabla}\cdot\vec{V}) \\
\therefore f_0 \vec{\nabla}\cdot\vec{V} &= -\frac{\partial \zeta_g}{\partial t} - \vec{V}_g \cdot \vec{\nabla}\zeta_g - v_g \frac{\partial f}{\partial y} \\
&= -\frac{\partial \zeta_g}{\partial t} - v\frac{\partial \zeta_g}{\partial s} - v_g \beta
\end{aligned}$$

$$\frac{\partial \zeta_g}{\partial t} = \frac{3\times 10^{-6}\,\mathrm{s}^{-1}}{(3\times 3600)\,\mathrm{s}} = 2.778\times 10^{-10}\,\mathrm{s}^{-2}$$

$$v\frac{\partial \zeta_g}{\partial s} = (20\,\mathrm{m\,s}^{-1})\left(\frac{4\times 10^{-6}\,\mathrm{s}^{-1}}{100\,000\,\mathrm{m}}\right) = 8\times 10^{-10}\,\mathrm{s}^{-2}$$

$$\begin{aligned}
(20\,\mathrm{m\,s}^{-1})^2 &= {u_g}^2 + {v_g}^2, \quad u_g = v_g \text{ (Pythagoras)} \\
\therefore v_g &= \pm\left(\frac{20^2}{2}\right)^{\frac{1}{2}} \\
\therefore v_g &= -14.14\,\mathrm{m\,s}^{-1} \quad (\text{since } v_g < 0) \\
\therefore v_g\beta &= -14.14\,\mathrm{m\,s}^{-1}(10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}) \\
&= -1.414\times 10^{-10}\,\mathrm{s}^{-2}
\end{aligned}$$

$$\begin{aligned}
\therefore \vec{\nabla}\cdot\vec{V} &= -{f_0}^{-1}\left(\frac{\partial \zeta_g}{\partial t} + v\frac{\partial \zeta_g}{\partial s} + v_g\beta\right) \\
&= -(-10^{-4}\,\mathrm{s}^{-1})^{-1}\left(2.778\times 10^{-10} + 8\times 10^{-10} - 1.414\times 10^{-10}\right)\mathrm{s}^{-2} \\
&= 9.364\times 10^{-6}\,\mathrm{s}^{-1}, \text{ divergence}
\end{aligned}$$

Exercise 2: Consider the following expression for the geopotential field:

$$\Phi = \Phi_0(p) + cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right] + k^{-1}\sin k(x - ct)\right\}$$

is a function of p alone, c is a constant speed, k a zonal wave number, and $p_0 = 1000\mathrm{hPa}$.

Consider the following two assumptions:

1. Only consider the dominating vorticity advection (either planetary or relative) term applicable to short-wave systems
2. Geostrophic relative vorticity only varies between trough and ridge axes in the x-direction

Use the quasi-geostrophic vorticity equation to show that the horizontal divergence field consistent with this geopotential field can be expressed as:

$$(f_0)^{-1}(ck)^2\cos\left(\frac{\pi p}{p_0}\right)\cos k(x - ct)$$

Solution: From Exercise 1: $\vec{\nabla} \cdot \vec{V} = -{f_0}^{-1} \left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla} \right) (\zeta_g + f)$

$$\begin{aligned} u_g &= -\frac{1}{f_0} \frac{\partial \Phi}{\partial y} \\ v_g &= \frac{1}{f_0} \frac{\partial \Phi}{\partial x} \\ \zeta_g &= \frac{1}{f_0} \nabla^2 \Phi \end{aligned}$$

$$\begin{aligned} -f_0 \vec{\nabla} \cdot \vec{V} &= \left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla} \right) (\zeta_g + f) = \frac{\partial \zeta_g}{\partial t} + \frac{\partial f}{\partial t} + \vec{V}_g \cdot \vec{\nabla} \zeta_g + \vec{V}_g \cdot \vec{\nabla} f \\ &= \frac{\partial \zeta_g}{\partial t} + (u_g \vec{i} + v_g \vec{j}) \cdot \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} \right) \zeta_g + (u_g \vec{i} + v_g \vec{j}) \cdot \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} \right) f \\ &= \frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} + v_g \frac{\partial f}{\partial y} \end{aligned}$$

We are considering short-wave systems, which means planetary vorticity advection is dominated by relative vorticity advection, thus $v_g \beta \sim 0$.

Also according to idealized 500hPa geopotential field, ζ_g only varies between trough and ridge axes in the x-direction, therefore $v_g \frac{\partial \zeta_g}{\partial y} = 0$.

$$\begin{aligned} \therefore -f_0 \vec{\nabla} \cdot \vec{V} &= \frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} \\ \therefore \vec{\nabla} \cdot \vec{V} &= -{f_0}^{-1} \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} \right) \zeta_g \end{aligned}$$

$$\begin{aligned} \zeta_g &= \frac{1}{f_0} \nabla^2 \Phi \\ \implies f_0 \zeta_g &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(c f_0 \left\{ -y \left[\cos\left(\frac{\pi p}{p_0} \right) + 1 \right] + \frac{1}{k} \sin k(x - ct) \right\} \right) \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(c f_0 \left\{ -y \left[\cos\left(\frac{\pi p}{p_0} \right) + 1 \right] + \frac{1}{k} \sin k(x - ct) \right\} \right) \right) \\ &= \frac{\partial}{\partial x} \left(c f_0 \frac{1}{k} k \cos k(x - ct) \right) \\ &= -c f_0 k \sin k(x - ct) \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \left(c f_0 \left\{ -y \cos\left(\frac{\pi p}{p_0} \right) - y + \frac{1}{k} \sin k(x - ct) \right\} \right) \right) \\ &= \frac{\partial}{\partial y} \left(c f_0 \left(-\cos\left(\frac{\pi p}{p_0} \right) - 1 \right) \right) \\ &= 0 \end{aligned}$$

$$f_0\zeta_g = -cf_0 k \sin k(x - ct)$$
$$\therefore \zeta_g = -ck \sin k(x - ct)$$

$$\begin{aligned}
u_g &= -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \\
\implies -f_0 u_g &= \frac{\partial}{\partial y}\left(cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right)+1\right]+\frac{1}{k}\sin k(x-ct)\right\}\right) \\
&= cf_0\left(-\left[\cos\left(\frac{\pi p}{p_0}\right)+1\right]\right) \\
\therefore u_g &= c\left(\cos\left(\frac{\pi p}{p_0}\right)+1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\zeta_g &= \frac{\partial}{\partial t}(-ck\sin k(x-ct)) \\
&= -ck(-ck\cos k(x-ct)) \\
&= c^2k^2\cos k(x-ct)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\zeta_g &= \frac{\partial}{\partial x}(-ck\sin k(x-ct)) \\
&= -ck(k\cos k(x-ct)) \\
&= -ck^2\cos k(x-ct)
\end{aligned}$$

$$\begin{aligned}
\therefore \vec{\nabla}\cdot\vec{V} &= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)+\left(c\left(\cos\left(\frac{\pi p}{p_0}\right)+1\right)\right)\times(-ck^2\cos k(x-ct))\right) \\
&= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)+\left(c\cos\left(\frac{\pi p}{p_0}\right)+c\right)\times(-ck^2\cos k(x-ct))\right) \\
&= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)-c^2k^2\cos\left(\frac{\pi p}{p_0}\right)\cos k(x-ct)-c^2k^2\cos k(x-ct)\right) \\
&= \frac{c^2k^2}{f_0}\cos\left(\frac{\pi p}{p_0}\right)\cos k(x-ct)
\end{aligned}$$

Exercise 3: Suppose that on the $500\mathrm{hPa}$ surface the relative vorticity at a location just left of the ridge line in the figure used in Exercise 1, at the $45°$S latitude (where the Coriolis parameter can be considered to be a constant value of $-10^{-4}\,\mathrm{s}^{-1}$ in the Southern Hemisphere) is increasing at a rate of $3.6\times10^{-6}\,\mathrm{s}^{-1}$ per hour. The wind is, for all practical purposes, blowing directly from the west above the location (negligible north-south component) at $20\,\mathrm{m\,s}^{-1}$ and the relative vorticity increases toward the east at a rate of $4\times10^{-6}\,\mathrm{s}^{-1}$ per $100\,\mathrm{km}$. Use the quasi-geostrophic vorticity equation to estimate the horizontal divergence at this location on a β-plane. This is a short-wave system.

Solution:

$$\begin{aligned}
\frac{\partial\zeta_g}{\partial t} &= -\vec{V}_g\cdot\vec{\nabla}(\zeta_g+f)+f_0\frac{\partial\omega}{\partial p} \\
&= -(u_g\vec{i}+v_g\vec{j})\cdot\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)(\zeta_g+f)+f_0\frac{\partial\omega}{\partial p}
\end{aligned}$$

Since we can ignore advection of planetary vorticity,

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} + f_0 \frac{\partial \omega}{\partial p}$$

Since the wind at the location is blowing from the west, $v_g = 0$

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} + f_0 \frac{\partial \omega}{\partial p}$$

$$\therefore \frac{\partial \omega}{\partial p} = f_0{}^{-1} \left(\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} \right)$$

$$\begin{aligned}
\frac{\partial \zeta_g}{\partial t} &> 0 \text{ (increasing at a rate of } 3.6 \times 10^{-6}\,\mathrm{s}^{-1}\text{)} \\
u_g &> 0 \text{ (wind from the west)} \\
\frac{\partial \zeta_g}{\partial x} &> 0 \text{ (vorticity increases per distance, and location is in Region I)}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{\partial \omega}{\partial p} &= \left(-10^{-4}\,\mathrm{s}^{-1}\right)^{-1} \left(\frac{3.6 \times 10^{-6}\,\mathrm{s}^{-1}}{1 \times 60 \times 60\,\mathrm{s}} + 20\,\mathrm{m\,s}^{-1}\, \frac{4 \times 10^{-6}\,\mathrm{s}^{-1}}{10^3\,\mathrm{m}} \right) \\
&= -10^4\,\mathrm{s} \left(\frac{36 \times 10^{-7}}{36 \times 10^2}\,\mathrm{s}^{-2} + 8 \times 10^{-10}\,\mathrm{s}^{-2} \right) \\
&= -10^4 \times \left(10^{-9}\,\mathrm{s}^{-1} + 8 \times 10^{-10}\,\mathrm{s}^{-1}\right) \\
&= -10^4 \times \left(10 \times 10^{-10} + 8 \times 10^{-10}\right)\mathrm{s}^{-1} \\
&= -1.8 \times 10^{-5}\,\mathrm{s}^{-1}, \text{ divergence since } -\frac{\partial \omega}{\partial p} = \vec{\nabla} \cdot \vec{V}
\end{aligned}$$

Quasi-geostrophic prediction

The geostrophic vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \tag{6.19}$$

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi \tag{6.15}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{1}{f_0} \nabla^2 \Phi \right) &= -\vec{V}_g \cdot \vec{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p} \\
\frac{1}{f_0} \nabla^2 \frac{\partial \Phi}{\partial t} &= -\vec{V}_g \cdot \vec{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p}
\end{aligned}$$

Defining the geopotential tendency $\chi \equiv \dfrac{\partial \Phi}{\partial t}$

$$\therefore \frac{1}{f_0} \nabla^2 \chi = -\vec{V}_g \cdot \vec{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p} \tag{6.21}$$

Since $\vec{V}_g = \dfrac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi$, the right-hand side of (6.21) depends only on the dependent variables Φ and ω.

Next, we will obtain an analogous equation also dependent on these two variables (Φ and ω)

Consider the thermodynamic energy equation:

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \vec{\nabla}\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \tag{6.13b}$$

$$\therefore \frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) + \vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

$$\therefore -\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) = \vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right) + \sigma\omega + \frac{\kappa J}{p}$$

$$\therefore \frac{\partial \chi}{\partial p} = -\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega - \frac{\kappa J}{p}$$

Multiply by f_0/σ: $\dfrac{f_0}{\sigma}\dfrac{\partial \chi}{\partial p} = -\dfrac{f_0}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\dfrac{\partial \Phi}{\partial p}\right) - f_0\omega - \dfrac{f_0 \kappa J}{\sigma p}$

Differentiate with respect to p:

$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)\right] - f_0\frac{\partial \omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right) \qquad \left[\sigma \equiv -\frac{RT_0}{p}\frac{d\ln\theta}{dp}\right] \tag{6.22}$$

The ageostrophic vertical motion, ω, has equal and opposite effects on the left-hand sides in $\left(6.21 : f_0^{-1}\nabla^2\chi\right)$ and $\left(6.22 : \dfrac{\partial}{\partial p}\left(\dfrac{f_0}{\sigma}\dfrac{\partial \chi}{\partial p}\right)\right)$

Vertical stretching $\left(\dfrac{\partial \omega}{\partial p} > 0\right)$ forces a positive tendency in the geostrophic vorticity (6.21) and a negative tendency of equal magnitude in the term on the left side in (6.22).

The left side of (6.22) can be interpreted as the local rate of change of a **normalized** static stability anomaly (i.e., a measure of the departure of static stability from S_p, its standard atmosphere value).

To demonstrate this statement:

$$\begin{aligned}
\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) &= \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)\right) = \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial p}\right)\right) \\
&= \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial t}\left(-\frac{RT}{p}\right)\right) \qquad \left\{(6.2) : \frac{\partial \Phi}{\partial p} = -\frac{RT}{p}\right\} \\
&= -f_0\frac{\partial}{\partial p}\left(\frac{R}{\sigma p}\frac{\partial T}{\partial t}\right) \qquad \left\{S_p = \frac{p\sigma}{R}\right\} \\
&= -f_0\frac{\partial}{\partial p}\left(\frac{1}{S_p}\frac{\partial T}{\partial t}\right) \\
&= -f_0\left[\frac{\partial}{\partial p}\left(\frac{1}{S_p}\right)\frac{\partial T}{\partial t} + \frac{1}{S_p}\frac{\partial}{\partial p}\frac{\partial T}{\partial t}\right]
\end{aligned}$$

Assume that S_p varies only slowly with height in the troposphere, thus S_p is nearly constant and $\frac{\partial}{\partial p}\left(S_p^{-1}\right) \approx 0$

$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) \approx -\frac{f_0}{S_p}\frac{\partial}{\partial p}\frac{\partial T}{\partial t} = -\frac{f_0}{S_p}\frac{\partial}{\partial t}\frac{\partial T}{\partial p} = -\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right)$$

From page 5 of the notes:

$$T_{tot} = T_0 + T \qquad \{T_0 : \text{basic state (standard atmosphere)}\}$$
$$\therefore T = T_{tot} - T_0$$

Therefore $\frac{\partial T}{\partial p} \sim$ local static stability anomaly

$$\therefore \frac{1}{S_p}\frac{\partial T}{\partial p} \sim \text{ Local static stability anomaly divided by the standard atmosphere static stability}$$

Take note: $\frac{f_0}{S_p}\frac{\partial T}{\partial p}$ has the same units as vorticity, and is also a **normalized** static stability value.

When the tendency of the normalized static stability anomaly > 0:

$$\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right) > 0$$
$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) < 0, \text{ the left side of (6.22)}$$

An air column that moves adiabatically from a region of high static stability to a region of low static stability, $\partial\omega/\partial p > 0$.

Since (6.21) and (6.22) are **analogous** equations, the relative vorticity in (6.21), $\frac{1}{f_0}\nabla^2\chi$ and the normalized static stability anomaly in (6.22) are changed by equal and opposite amounts. The normalized static stability anomaly is therefore referred to as the stretching vorticity.

Purely geostrophic motion ($\omega = 0$) is a solution to (6.21) and (6.22) only in a very special situations such as barotropic flow (no pressure dependence) or zonally symmetric flow. More general purely geostrophic flows cannot satisfy both these equations simultaneously as there are then two independent equations, and a single unknown (Φ) so that the system is overdetermined. Thus, the role of the vertical motion distribution must be to maintain consistency between the geopotential tendencies required by vorticity advection in (6.21) and thermal advection in (6.22).

Geopotential tendency

(6.21):

$$\therefore \frac{1}{f_0}\nabla^2\chi = -\vec{V}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p} \tag{1}$$

Assuming that the diabatic heating rate $J = 0$, (6.22) becomes:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right] - f_0\frac{\partial\omega}{\partial p} \tag{2}$$

$(1) + (2)$:

$$\left[\frac{1}{f_0}\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -\vec{V}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]$$

$$\therefore \underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{\text{A}} = \underbrace{-f_0\vec{V}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{B}} \underbrace{-\frac{\partial}{\partial p}\left[-\frac{{f_0}^2}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\text{C}} \tag{6.23}$$

(6.23) is often referred to as the **geopotential tendency equation**.

A. The local geopotential tendency

B. The distribution of vorticity advection

C. The thickness advection

If the distribution of Φ is known at a given time, B and C may be regarded as known forcing functions and (6.23) is a linear partial differential equation in the unknown χ.

Take note that the term A involves second derivatives in space (x, y) of the field χ, and thus generally proportional to $-\chi$.

$\chi = \dfrac{\partial\Phi}{\partial t}$ and we assume that the horizontal structure of Φ (geopotential) in the extra-tropics can be represented by a sinusoidal function:

$$\Phi = \Phi(x, y, p, t) = A(p, t)B(x, y)$$

$$\text{with } B(x, y) = \sin(kx)\cos(ly);\ k = \frac{2\pi}{L_x};\ l = \frac{2\pi}{L_y}$$

$$\therefore \chi = \frac{\partial}{\partial t}\left(A(p, t)\sin(kx)\cos(ly)\right)$$

Term A: $\left[\nabla^2 + \dfrac{\partial}{\partial p}\left(\dfrac{{f_0}^2}{\sigma}\dfrac{\partial}{\partial p}\right)\right]$ applied to χ:

$$\nabla^2\chi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\frac{\partial A}{\partial t}\sin(kx)\cos(ly)\right]$$
$$= \frac{\partial A}{\partial t}\left(\frac{\partial^2}{\partial x^2}(\sin(kx)\cos(ly)) + \frac{\partial^2}{\partial y^2}(\sin(kx)\cos(ly))\right)$$
$$= \frac{\partial A}{\partial t}\left(\cos(ly)\frac{\partial^2}{\partial x^2}(\sin(kx)) + \sin(kx)\frac{\partial^2}{\partial y^2}(\cos(ly))\right)$$

$$\frac{\partial^2}{\partial x^2}(\sin(kx)) = \frac{\partial}{\partial x}(k\cos(kx)) = -k^2\sin(kx)$$
$$\frac{\partial^2}{\partial y^2}(\cos(ly)) = \frac{\partial}{\partial y}(-l\sin(ly)) = -l^2\cos(ly)$$

$$\therefore \nabla^2\chi = \frac{\partial A}{\partial t}\left(\cos(ly)(-k^2\sin(kx)) + \sin(kx)(-l^2\cos(ly))\right)$$
$$= \frac{\partial A}{\partial t}\sin(kx))\cos(ly)(-k^2 - l^2)$$
$$= -(k^2 + l^2)\frac{\partial A}{\partial t}\sin(kx))\cos(ly)$$
$$= -(k^2 + l^2)\chi \propto -\chi$$

Since geopotential fields tend to lean **westward** with height in the mid-latitudes an upper troposphere ridge often lies over or near the surface trough:

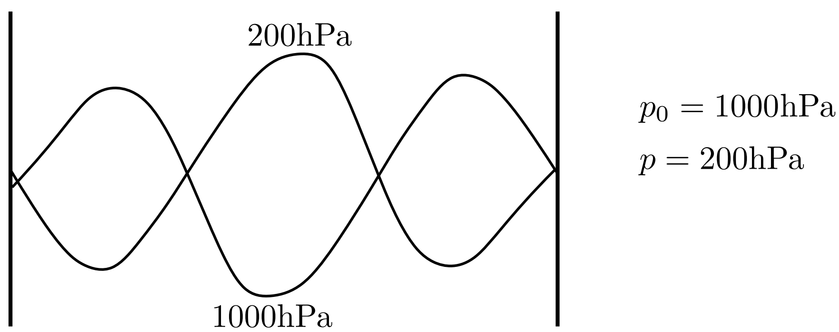


Figure 14: A full phase shift with height.

$\Phi = A(p,t)B(x,y)$; we dealt with the $B(x,y)$ part on the previous page, and so we now consider

$$A(p,t) = Q(t)\cos\left(\frac{\pi p}{p_0}\right)$$

$$\chi = \frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}\left(Q(t)\cos\left(\frac{\pi p}{p_0}\right)B(x,y)\right)$$
$$= \cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}$$

Regarding term A of the geopotential tendency equation, apply $\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)$ to χ:

$$\begin{aligned}
\therefore \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\left(\cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}\right) \\
&= B\frac{\partial Q}{\partial t}\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\left(\cos\left(\frac{\pi p}{p_0}\right)\right) \\
&= B\frac{\partial Q}{\partial t}\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\left(\frac{-\pi}{p_0}\right)\sin\left(\frac{\pi p}{p_0}\right)\right)\right]
\end{aligned}$$

Assume that the standard atmosphere static stability parameter σ, varies only slowly with height $\left(\text{i.e., } \frac{\partial}{\partial p}(\sigma^{-1}) \approx 0\right)$ in the troposphere:

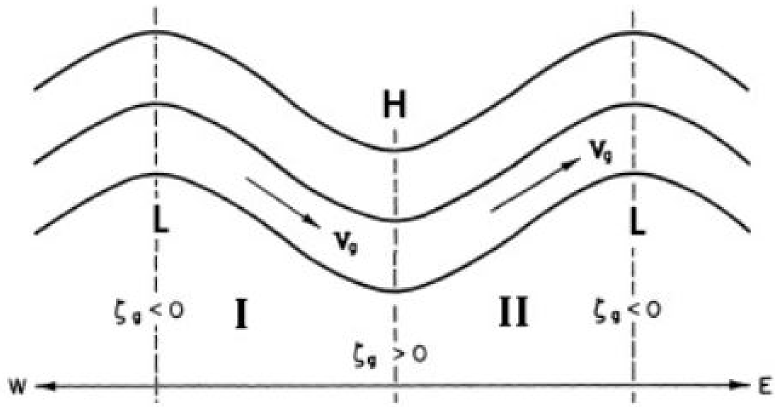
$$\begin{aligned}
\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= -B\frac{\partial Q}{\partial t}\frac{{f_0}^2}{\sigma}\frac{\pi}{p_0}\frac{\partial}{\partial p}\left(\sin\left(\frac{\pi p}{p_0}\right)\right) \\
\therefore \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= -B\frac{\partial Q}{\partial t}\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\cos\left(\frac{\pi p}{p_0}\right) \\
&= -\cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}\left(\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\right) \\
&= -\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\chi \propto -\chi \\
&\implies \left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \propto -\chi
\end{aligned}$$

Term A is thus generally proportional to $-\chi\left(=\frac{\partial \Phi}{\partial t}\right)$

Next, consider Term B:

$$\begin{aligned}
-f_0\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) &= -f_0\vec{V}_g\cdot\vec{\nabla}\left(\zeta_g + f\right) \qquad \left[(6.15) : \zeta_g = \frac{1}{f_0}\nabla^2\Phi\right] \\
&= -f_0\vec{V}_g\cdot\vec{\nabla}\zeta_g - f_0\vec{V}_g\cdot\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right) \\
&= -f_0\vec{V}_g\cdot\vec{\nabla}\zeta_g - f_0(u_g\vec{i} + v_g\vec{j})\cdot\frac{\partial f}{\partial y}\vec{j} \qquad (f \neq f(x)) \\
&= -f_0\vec{V}_g\cdot\vec{\nabla}\zeta_g - f_0 v_g\frac{\partial f}{\partial y} \\
&= \text{geostrophic advection of relative vorticity} + \\
&\qquad \text{geostrophic advection of planetary vorticity}
\end{aligned}$$

Consider the schematic below of a 500hPa geopotential field in the **Southern Hemisphere**:



Region I: Upstream of the 500hPa ridge, the geostrophic wind is directed from the relative vorticity minimum at the trough towards the relative vorticity maximum at the ridge.

$\Longrightarrow$ Advection of relative vorticity is positive.

$$\therefore \vec{V}_g \cdot \vec{\nabla}\zeta_g > 0$$
$$\therefore f_0\vec{V}_g \cdot \vec{\nabla}\zeta_g < 0 \qquad \text{in the SH } (f_0 < 0)$$
$$\therefore -f_0\vec{V}_g \cdot \vec{\nabla}\zeta_g > 0 \qquad \text{in the SH}$$

At the same time $v_g < 0$ because it is directed southwards, and $\frac{\partial f}{\partial y} = \beta = 2\Omega \cos\phi_0/a > 0$ (both hemispheres).

$$\therefore v_g\frac{\partial f}{\partial y} < 0$$
$$\therefore f_0 v_g\frac{\partial f}{\partial y} > 0 \qquad \text{in the SH } (f_0 < 0)$$
$$\therefore -f_0 v_g\frac{\partial f}{\partial y} < 0 \qquad \text{in the SH}$$

For advection of relative vorticity:

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \propto -\chi > 0$$
$$\therefore \chi < 0$$
$$\therefore \frac{\partial \Phi}{\partial t} < 0$$

therefore the geopotential heights are falling between the trough and the ridge axis, downstream of the trough axis.

For advection of planetary vorticity:

$$
\begin{aligned}
&-\chi < 0 \qquad \left[-f_0 v_g \frac{\partial f}{\partial y} < 0\right] \\
&\therefore \chi > 0 \\
&\therefore \frac{\partial \Phi}{\partial t} > 0
\end{aligned}
$$

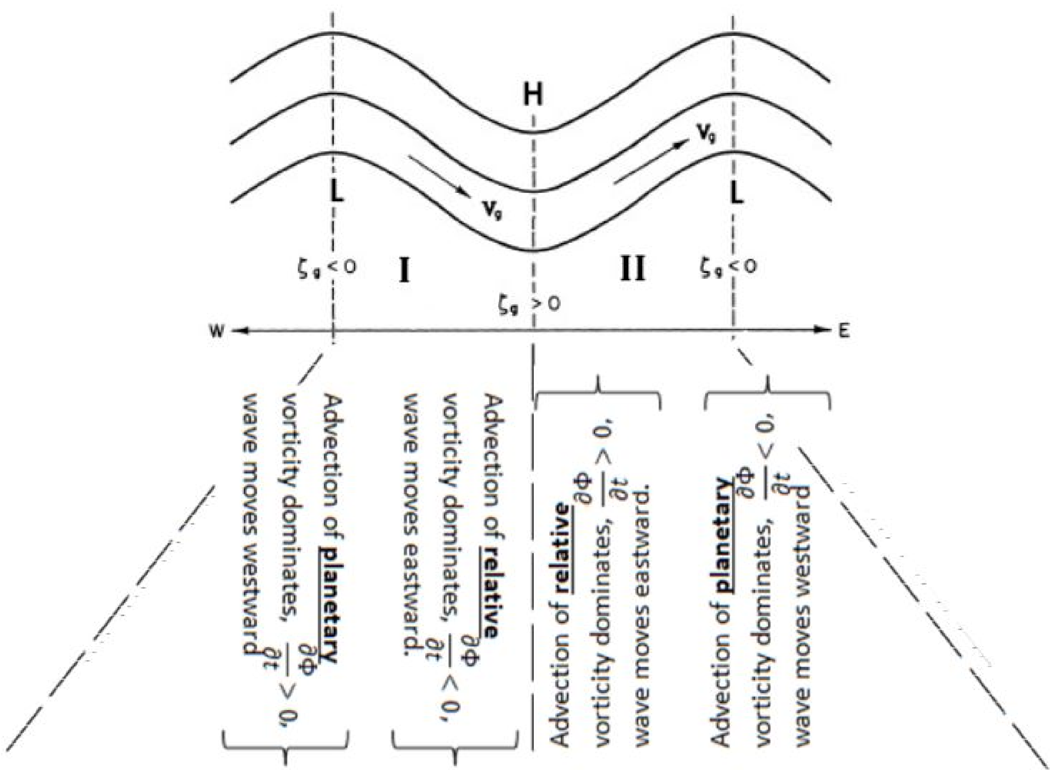
which implies that the advection of planetary vorticity results in increasing geopotential heights.

Similarly for **Region II**:

$$
\begin{aligned}
-f_0 \vec{V}_g \cdot \vec{\nabla} \zeta_g &< 0 \qquad \text{in the SH} \\
\therefore -\chi &< 0 \\
\therefore \chi &> 0 \\
\therefore \frac{\partial \Phi}{\partial t} &> 0
\end{aligned}
$$

and

$$
\begin{aligned}
-f_0 v_g \frac{\partial f}{\partial y} &> 0 \\
\therefore -\chi &> 0 \\
\therefore \chi &< 0 \\
\therefore \frac{\partial \Phi}{\partial t} &< 0
\end{aligned}
$$



For a mid-latitude disturbance of given amplitude the absolute value of the relative vorticity **increases** for **decreasing** wavelength.

Therefore for short wavelengths ($\leq 3000\mathrm{km}$) the advection of relative vorticity tends to dominate, resulting in the disturbance moving rapidly eastwards.

For long waves ($\geq 10000\mathrm{km}$) the planetary vorticity advection tends to dominate, resulting in these long planetary waves to be quasi-stationary.

Since $\vec{\nabla}\zeta_g$ and v_g are zero at both trough and ridge axes, the vorticity advection term is zero:

$$\begin{aligned}\text{Term B:} &- f_0\vec{V}_g \cdot \vec{\nabla}\zeta_g - f_0 v_g \frac{\partial f}{\partial y} \\ &= -f_0\vec{V}_g \cdot \vec{0} - f_0(0)\frac{\partial f}{\partial y} \\ &= 0\end{aligned}$$

$\implies$ Vorticity advection cannot change the strength of this type of disturbance at the levels where the advection is occurring, but only acts to propagate the disturbance horizontally and (as shown in the next section) to spread it vertically.

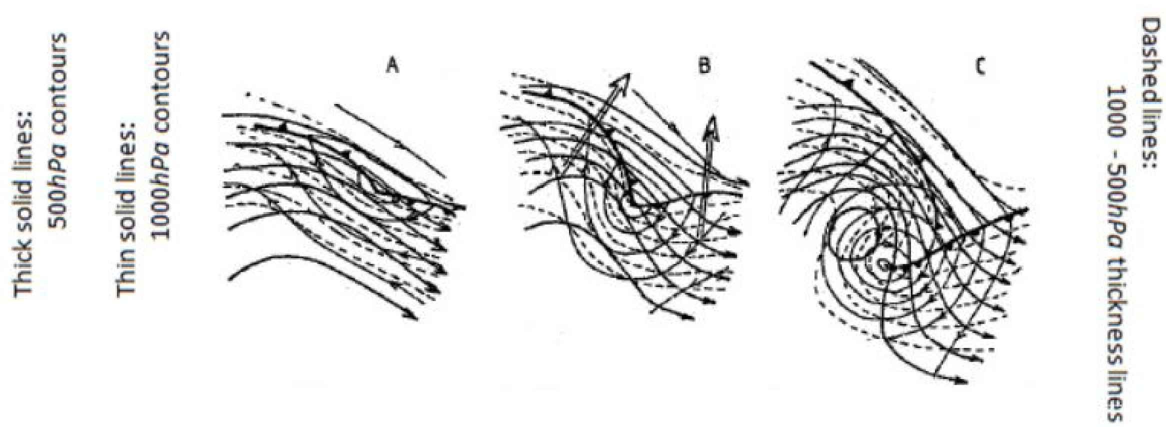
The mechanism for amplification or decay of mid-latitude synoptic systems is contained in Term C:

$$-\frac{\partial}{\partial p}\left[-\frac{{f_0}^2}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] = \frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

This term is called the differential thickness advection and it tends to be a maximum at trough and ridge lines in a developing baroclinic wave.

The term $\vec{V}_g{\cdot}\vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is proportional to the hydrostatic temperature advection, and $\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$ is proportional to the rate of change of the temperature advection with height, or the **differential temperature advection**.

Consider below an idealized schematic representation of a developing baroclinic disturbance:



In order to determine the rate of change of the temperature advection with height (or pressure) at least two levels in the vertical must be used. Here two layers are considered: $1000\text{–}500\mathrm{hPa}$ layer (lower troposphere), and the $500\text{–}300\mathrm{hPa}$ layer (upper troposphere).

The figure above demonstrates that a developing baroclinic disturbance is characterized by the westward tilt with height of the pressure system (the thick solid contours are to the west of the thin contours).

1. The tilting results in strong cold advection behind the cold front and strong warm advection ahead of the warm front.
2. In the upper troposphere, the tilt of the pressure system is small.

These statements are demonstrated in the figure below.

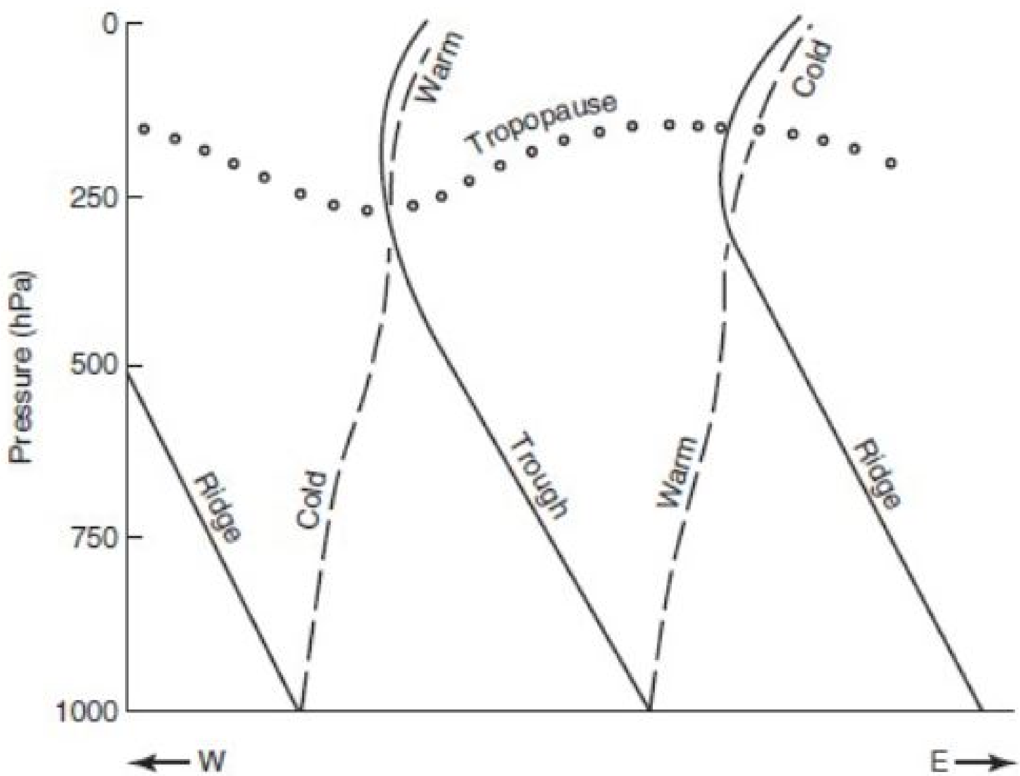


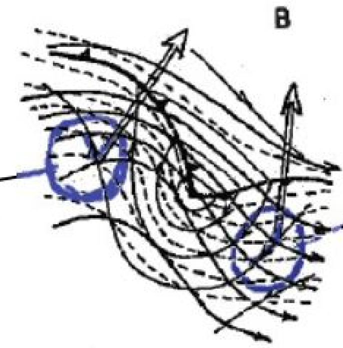
Figure 15: West-east cross section through a developing baroclinic wave. Solid lines are trough and ridge axes; dashed lines are axes of temperature extrema; the chain of open circle denotes the tropopause.

In the upper troposphere the tilt of the pressure system with height in small. The result is that the thickness pattern and the geopotential pattern become approximately parallel, which leads to thermal advection becoming small there. Term C is thus concentrated in the lower troposphere.

For the case of the lower troposphere, we want to determine the sign and the magnitude of the Term C. The horizontal thermal advection for this part of the troposphere ($1000\text{–}500\mathrm{hPa}$ layer) is given by:

$$\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$$

where $\vec{V}_g$ is the geostrophic wind at the $1000\mathrm{hPa}$ level, and $\vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is a vector that is perpendicular to the 1000– 500hPa thickness lines. See diagram B on page 124. This vector points towards the warm sector of the low pressure system, and is shown in B for two positions (the two arrows).



The sign of the scalar product of $\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is given by:

$$\left|\vec{V}_g\right| \left|\vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right| \cos\theta$$

$$\cos\theta > 0 \text{ if } \theta < 90°\text{, behind cold front}$$
$$\cos\theta < 0 \text{ if } \theta > 90°\text{, ahead of warm front}$$

Behind cold front (below 500hPa trough):

$$\text{Thermal advection} = \vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0, \text{ cold advection}$$

Ahead of warm front (below 500hPa ridge):

$$\text{Thermal advection} = \vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) < 0, \text{ warm advection}$$

Recalling the discussion above that for a developing system the thermal advection is much smaller in the upper troposphere than in the lower troposphere, thermal advection (both cold and warm) decreases with height. So does tropospheric pressure.

$$\therefore \frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right) > 0 \quad \text{below } 500\text{hPa trough}$$
$$\text{and} \quad \frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right) < 0 \quad \text{below } 500\text{hPa ridge}$$

Since $\frac{{f_0}^2}{\sigma}$:

$$\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] \begin{cases} < 0 & \text{at ridge} \\ > 0 & \text{at trough} \end{cases}$$

$$
\begin{cases}
\text{At the ridge:} & \dfrac{{f_0}^2}{\sigma}\dfrac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(-\dfrac{\partial \Phi}{\partial p}\right)\right] < 0 \quad \text{(warm advection)} \\
& \therefore -\chi < 0 \\
& \therefore \chi > 0 \\
& \therefore \dfrac{\partial \Phi}{\partial t} > 0 \quad \text{(geopotential increases with time)}
\end{cases} \qquad (*)
$$

$$
\begin{cases}
\text{At the trough:} & \dfrac{{f_0}^2}{\sigma}\dfrac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(-\dfrac{\partial \Phi}{\partial p}\right)\right] > 0 \quad \text{(cold advection)} \\
& \therefore -\chi > 0 \\
& \therefore \chi < 0 \\
& \therefore \dfrac{\partial \Phi}{\partial t} < 0 \quad \text{(geopotential decreases with time)}
\end{cases} \qquad (+)
$$

(*) The effect of warm advection below the 500hPa ridge is to build the ridge.

$(+)$ The effect of cold advection below the 500hPa trough is to deepen the trough.

$\Longrightarrow$ The differential temperature or thickness advection intensifies the upper level troughs and ridges in a developing baroclinic system.

The advection of cold air into the air column below the 500hPa trough will reduce the thickness of that column, and hence will lower the height of the 500hPa surface unless there is a compensating rise in the surface pressure. Warm advection into the air column below the 500hPa ridge will have the opposite effect.

The traditional omega equation

The vorticity equation (6.19):

$$
\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p}
$$

ζ_g and $\vec{V}_g$ are both defined in terms of $\Phi(x, y, p, t)$:

$$
\zeta_g = \frac{1}{f_0}\nabla^2 \Phi \quad \text{and} \quad \vec{V}_g = \frac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi
$$

Therefore the vorticity equation (6.19) can be used to diagnose ω (vertical velocity field) provided that the fields of both Φ and $\dfrac{\partial \Phi}{\partial t}$ are known.

Φ : primary product of operational weather analysis

$\dfrac{\partial \Phi}{\partial t}$: can only be crudely approximated from observations by taking differences over 12 hours, since upper level analyses are generally available only twice per day.

Despite this limitation, the vorticity equation method of estimating ω is usually more accurate than the continuity equation method discussed in WKD352 (the kinematic method). However, neither of these two methods of estimating ω uses the information available in the thermodynamic energy equation. Here we will develop the so-called omega equation for estimating the vertical motion by utilizing both the vorticity equation and the thermodynamic equation.

Thermodynamic energy equation (6.13b):

$$\left(\frac{\partial}{\partial t}+\vec{V}_g\cdot\vec{\nabla}\right)\left(-\frac{\partial\Phi}{\partial p}\right)-\sigma\omega=\frac{\kappa J}{p}$$

Apply the horizontal Laplacian:

$$\begin{aligned}
&\nabla^2\left(\frac{\partial}{\partial t}+\vec{V}_g\cdot\vec{\nabla}\right)\left(-\frac{\partial\Phi}{\partial p}\right)-\nabla^2(\sigma\omega)=\nabla^2\left(\frac{\kappa J}{p}\right)\\
&\therefore\nabla^2\frac{\partial}{\partial t}\left(-\frac{\partial\Phi}{\partial p}\right)=\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]+\sigma\nabla^2\omega+\frac{\kappa}{p}\nabla^2 J\\
&\therefore\nabla^2\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right)=-\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\sigma\nabla^2\omega-\frac{\kappa}{p}\nabla^2 J\\
&\therefore\nabla^2\frac{\partial\chi}{\partial p}=-\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\sigma\nabla^2\omega-\frac{\kappa}{p}\nabla^2 J \qquad (6.32)
\end{aligned}$$

Rewriting the geostrophic vorticity equation:

$$\frac{1}{f_0}\nabla^2\chi=-\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)+f_0\frac{\partial\omega}{\partial p} \qquad (6.21)$$

Differentiate (6.21) with respect to p:

$$\begin{aligned}
&\therefore\frac{\partial}{\partial p}\left(\frac{1}{f_0}\nabla^2\chi\right)=-\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+f_0\frac{\partial^2\omega}{\partial p^2}\\
&\therefore\frac{\partial}{\partial p}\left(\nabla^2\chi\right)=-f_0\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+{f_0}^2\frac{\partial^2\omega}{\partial p^2} \qquad (6.33)
\end{aligned}$$

(6.33) – (6.32):

$$\begin{aligned}
&\left({f_0}^2\frac{\partial^2}{\partial p^2}+\sigma\nabla^2\right)\omega-f_0\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]+\frac{\kappa}{p}\nabla^2 J\\
&=\frac{\partial}{\partial p}\left(\nabla^2\chi\right)-\nabla^2\frac{\partial\chi}{\partial p}
\end{aligned}$$

Since the operators on the right hand side can be reversed:

$$\begin{aligned}
&\left({f_0}^2\frac{\partial^2}{\partial p^2}+\sigma\nabla^2\right)\omega=f_0\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]-\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\frac{\kappa}{p}\nabla^2 J\\
&\therefore\left(\nabla^2+\frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega=\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+\frac{1}{\sigma}\nabla^2\left[\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]-\frac{\kappa}{\sigma p}\nabla^2 J \qquad (6.34)
\end{aligned}$$

Term A : $\left(\nabla^2+\frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega$

Term B : $\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]=\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g\cdot\vec{\nabla}\left(\zeta_g+f\right)\right]$

Term C $: \dfrac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\dfrac{\partial\Phi}{\partial p}\right)\right]$

Term D $: -\dfrac{\kappa}{\sigma p}\nabla^2 J$, but as with the geopotential tendency equation we set $J = 0$; J is the diabatic heat rate.

The resulting omega equation:

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

The omega equation above involves only derivatives in space (not time). This equation is thus a **diagnostic equation** for the field of omega (ω) in terms of the instantaneous geopotential (Φ) field.

Remember the operator in Term A of the tendency equation? It is $\left(\nabla^2 + \dfrac{\partial}{\partial p}\left(\dfrac{{f_0}^2}{\sigma}\dfrac{\partial}{\partial p}\right)\right)$, and is very similar to the operator of Term A of the omega equation.

The forcing in the omega equation tends to be a maximum in the mid-troposhpere (500hPa), and ω is required to be zero at the surface and at the top of the troposphere. Therefore, for a qualitative discussion it is permissible to assume that ω has sinusoidal behaviour in both the horizontal and vertical:

$$\omega = W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly)$$

$$\begin{aligned}
\therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\left(W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly)\right) \\
&= \frac{\partial^2}{\partial x^2}\sin(kx)\left[W_0\sin\left(\frac{\pi p}{p_0}\right)\sin(ly)\right] \\
&+ \frac{\partial^2}{\partial y^2}\sin(ly)\left[W_0\sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\right] \\
&+ \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\sin\left(\frac{\pi p}{p_0}\right)\left[W_0\sin(kx)\sin(ly)\right] \\
&= -k^2\sin(kx)\left[W_0\sin\left(\frac{\pi p}{p_0}\right)\sin(ly)\right] \\
&- l^2\sin(ly)\left[W_0\sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\right] \\
&- \frac{{f_0}^2}{\sigma}\left(\frac{\pi}{p_0}\right)^2\sin\left(\frac{\pi p}{p_0}\right)\left[W_0\sin(kx)\sin(ly)\right] \\
&= W_0\sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly)\left[-k^2 - l^2 - \frac{1}{\sigma}\left(\frac{\pi f_0}{p_0}\right)^2\right] \\
&= -\left[k^2 + l^2 + \frac{1}{\sigma}\left(\frac{f_0\pi}{p_0}\right)^2\right]\omega
\end{aligned}$$

$\therefore$ Term A is proportional to $-\omega$

For synoptic-scale motions $\quad \omega = -\rho g w$

$\therefore \omega \propto -w$

$\therefore \omega < 0$ implies upward vertical motion.

Since $\omega < 0$ implies upward motion, and

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$$

$$\therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto w, \text{ the vertical velocity}$$

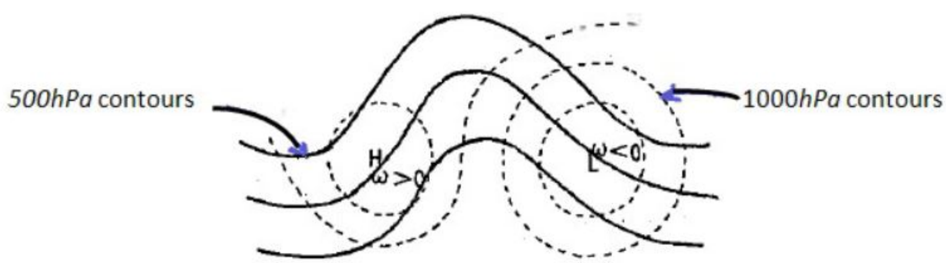
$\implies$ Upward motion is forced where the right-hand side of the omega equation is positive and downward motion is forced where it is negative.

The omega equation with negligible diabatic heating:

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

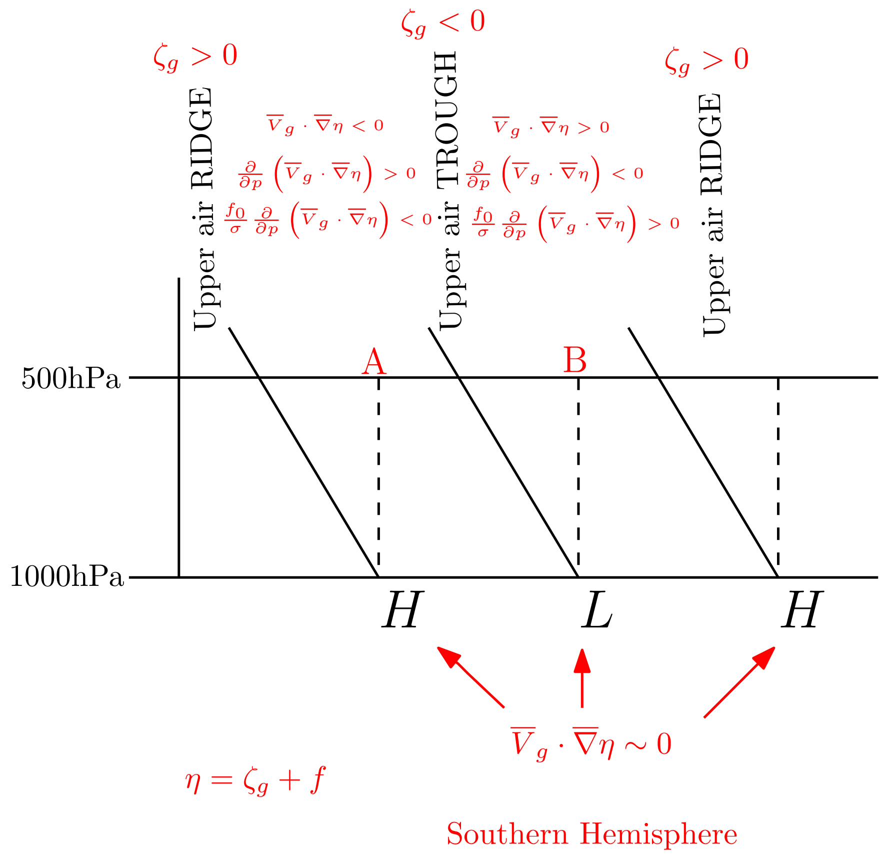
Term B: $\dfrac{f_0}{\sigma}\dfrac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g + f\right)\right]$, the differential vorticity advection.

This term is proportional to the rate of increase with height, or with pressure, of the advection of absolute vorticity. To discuss the role of this term we consider an idealized developing baroclinic system. Moreover, we consider a short-wave system where **<u>relative vorticity advection is larger than the planetary vorticity advection</u>**. The figure below shows schematically the geopotential contours at $500\mathrm{hPa}$ and $1000\mathrm{hPa}$ for such a system.



H: centre of the surface high pressure system
L: centre of the surface low pressure system

At the centres of the surface high and surface low $\vec{\nabla}\zeta_g$ and $\vec{V}_g$ must be very small: Previously it was discussed that, since $\vec{\nabla}\zeta_g$ and $\vec{V}_g$ are zero at both trough and ridge axes, the vorticity advection term is zero at the axes. However, since the H and L centres are not located exactly on the $500\mathrm{hPa}$ trough/ridge axis, $\vec{\nabla}\zeta_g$ and $\vec{V}_g$ are only very small (not zero) and so vorticity advection must be very small, that is $\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g + f\right)$ must be very small.



At point A, $\vec{V}_g \cdot \vec{\nabla}\eta < 0$ since the flow is going from a ridge where $\zeta_g > 0$ towards a trough where $\zeta_g < 0$. From the 500hPa level towards the surface where the high is $\left(\vec{V}_g \cdot \vec{\nabla}\eta \sim 0\right)$, there is thus an **increase** in $\vec{V}_g \cdot \vec{\nabla}\eta$ along the vertical pressure axis. Therefore

$$\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\eta\right) > 0$$
$$\therefore \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\eta\right) < 0 \quad \text{above the surface high}$$

At point B, $\vec{V}_g \cdot \vec{\nabla}\eta > 0$ since the flow is going from a trough where $\zeta_g < 0$ towards a ridge where $\zeta_g > 0$. From the 500hPa level towards the surface where the low is $\left(\vec{V}_g \cdot \vec{\nabla}\eta \sim 0\right)$, there is thus a **decrease** in $\vec{V}_g \cdot \vec{\nabla}\eta$ along the vertical pressure axis. Therefore

$$\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\eta\right) < 0$$
$$\therefore \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\eta\right) > 0 \quad \text{above the surface low}$$

Considering that $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto w$, and that Term B > 0 above point L, $w > 0$, that means **ascending motion**.

Since Term B < 0 above point H, $w < 0$, that means **subsiding motion**. Now for Term C: $\frac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$, and remembering that

$$\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0 \text{ for cold advection}$$

and

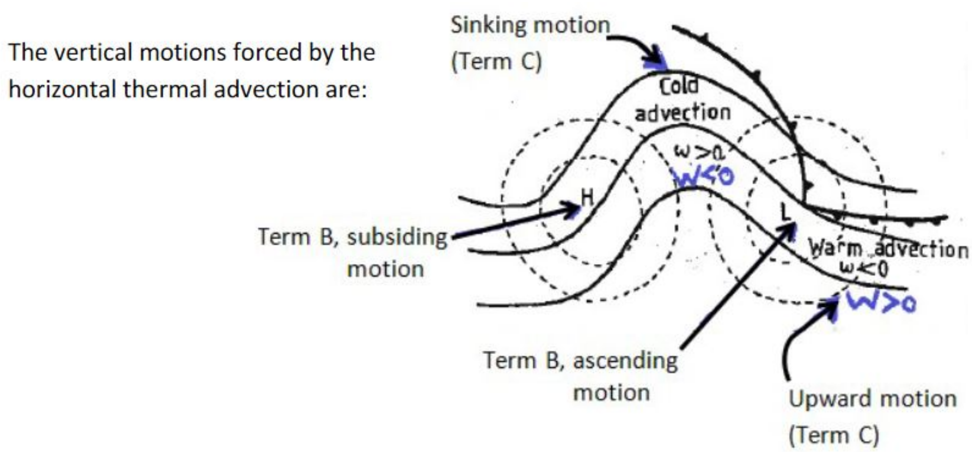
$$\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) < 0 \text{ for warm advection}$$

Consider the diagram at the top of Page 126: East of the surface low, in the warm front zone, the warm advection tends to be a maximum and west of the surface low, behind the cold front, the cold advection tends to be a maximum.

East of surface low: Warm advection, $\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) < 0$

However, we have shown already (twice!) that $\nabla^2 Y \propto -Y$

$$\begin{aligned}
\therefore \text{ East of surface low} \quad & \nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0 \\
\therefore \frac{1}{\sigma}& \nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0 \\
& w > 0 \text{ and maximum} \\
\text{West of surface low} \quad & \nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0 \\
\therefore \frac{1}{\sigma}& \nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0 \\
& w < 0 \text{ and minimum}
\end{aligned}$$



Bonus homework: Write a short essay (not more than one page) on the so-called Dines compensation.

The Sutcliffe form of the omega equation

A problem with the traditional omega equation is that there exists significant cancellation between the two terms on the right hand side of this form of the equation. Here we are presenting an alternative approximate form of the omega equation that can be applied in synoptic analysis in the Southern Hemisphere.

First, employ the chain rule of differentiation for the two terms on the right hand side of the traditional omega equation.

The omega equation:

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g + f\right)\right]}_{\text{B}} + \underbrace{\frac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{C}}$$

Apply the chain rule of differentiation on Term B:

$$\begin{aligned}
&\frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + \vec{V}_g \cdot \left(\frac{1}{f_0}\frac{\partial \nabla^2 \Phi}{\partial p} + \frac{\partial f}{\partial p}\right)\right] \\
&= \frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + \frac{1}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \nabla^2 \Phi}{\partial p}\right) \qquad (6.35a)
\end{aligned}$$

Also for Term C:

$$\begin{aligned}
&\frac{1}{\sigma}\left(\nabla^2 \vec{V}_g\right) \cdot \left(-\frac{\partial \Phi}{\partial p}\right) + \frac{1}{\sigma}\vec{V}_g \cdot \nabla^2\left(\vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right) \\
&= -\frac{1}{\sigma}\left(\left(\nabla^2 u_g\right)\vec{i} + \left(\nabla^2 v_g\right)\vec{j}\right) \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\left(\frac{\partial \Phi}{\partial p}\right) - \frac{1}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \nabla^2 \Phi}{\partial p}\right) \\
&= -\frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right)\right] - \frac{1}{\sigma}\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \nabla^2 \Phi}{\partial p}\right) \qquad (6.35b)
\end{aligned}$$

<u>NOTE</u>: The last terms in (6.35a) and (6.35b) are equal and opposite, therefore they cancel.

$$\begin{aligned}
\therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega &= \text{Term B} + \text{Term C} \\
&= \frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \qquad (\text{B1}) \\
&\quad - \frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right)\right] \qquad (\text{C1})
\end{aligned}$$

Scale analysis of these two expanded terms can help to compare the relative sizes of the two terms in order to reduce them.

$$\underline{\textbf{Note}}: \quad R = 287\,\mathrm{J\,K^{-1}\,kg^{-1}}$$
$$1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$$
$$1\,\mathrm{J} = 1\,\mathrm{N\,m}$$

Term B1: $\frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p}\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] = \frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p}\cdot\vec{\nabla}\left(\zeta_g + f\right)\right]$

$$\begin{aligned}\frac{\partial \vec{V}_g}{\partial p} \sim \frac{10\,\mathrm{m\,s^{-1}}}{10\times 10^2\,\mathrm{Pa}} &= \frac{1\,\mathrm{m\,s^{-1}}}{10^2\,\mathrm{N\,m^{-2}}}\\ &= 10^{-2}\,\mathrm{N^{-1}\,m^3\,s^{-1}}\\ &= 10^{-2}\,\mathrm{kg^{-1}\,m^{-1}\,s^2\,m^3\,s^{-1}}\\ &= 10^{-2}\,\mathrm{kg^{-1}\,m^2\,s}\end{aligned}$$

$$\begin{aligned}\vec{\nabla}\left(\zeta_g + f\right) &\sim \frac{1}{L}\left(10^{-5} - 10^{-4}\right)\mathrm{s^{-1}}\\ &\sim 10^{-6}\,\mathrm{m^{-1}}\left(10^{-4}\right)\mathrm{s^{-1}}\\ &= 10^{-10}\,\mathrm{m^{-1}\,s^{-1}}\end{aligned}$$

$$\begin{aligned}\therefore B1 &\sim \frac{10^{-4}\,\mathrm{s^{-1}}}{\sigma}\left[10^{-2}\,\mathrm{kg^{-1}\,m^2\,s}\right]\left[10^{-10}\,\mathrm{m^{-1}\,s^{-1}}\right]\\ &= \frac{1}{\sigma}10^{-16}\,\mathrm{kg^{-1}\,m\,s^{-1}}\end{aligned}$$

Term C1: $-\frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial p}\right)\right]$

$$\begin{aligned}\frac{\partial\Phi}{\partial p} = -\frac{RT}{p} &\sim \frac{10^2\,\mathrm{J\,K^{-1}\,kg^{-1}}\left(10^2\,\mathrm{K}\right)}{1000\times 10^2\,\mathrm{Pa}}\\ &= \frac{10^4\,\mathrm{N\,m\,kg^{-1}}}{10^5\,\mathrm{N\,m^{-2}}} = 10^{-1}\,\mathrm{m^3\,kg^{-1}}\end{aligned}$$

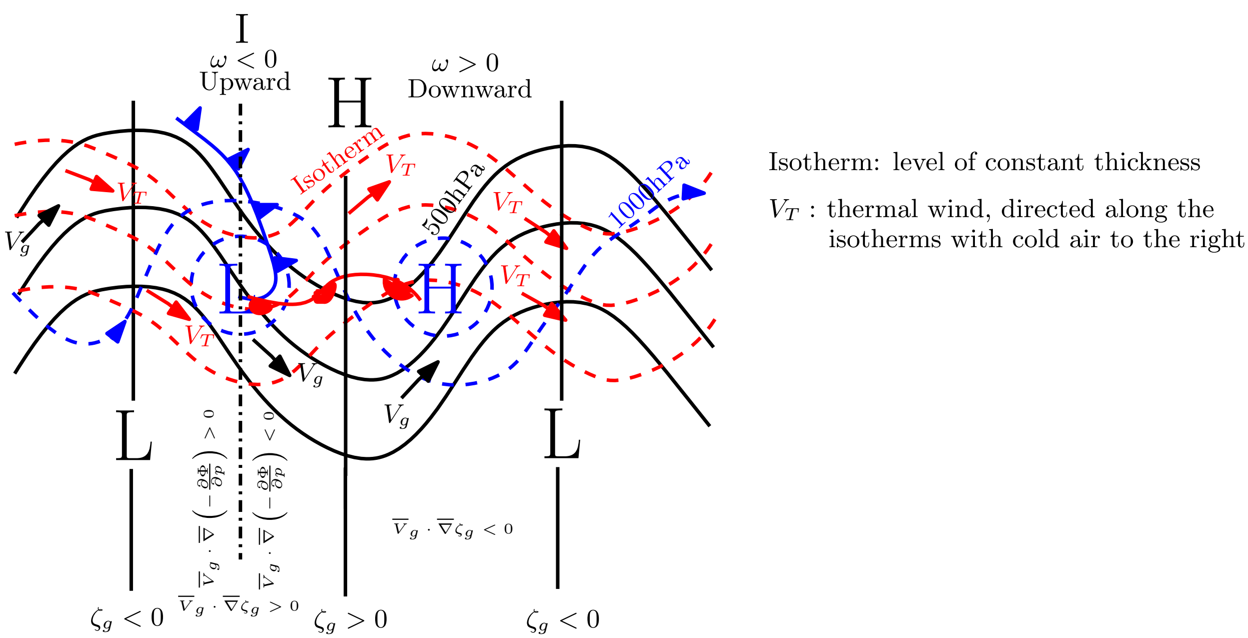
$$\begin{aligned}\therefore C1 \sim \frac{1}{\sigma}\left(\frac{1}{L^2}U\frac{1}{L}\left(10^{-1}\,\mathrm{m^3\,kg^{-1}}\right)\right) &\sim \frac{1}{\sigma}\left(10^6\,\mathrm{m}\right)^{-3}10\,\mathrm{m\,s^{-1}}\,10^{-1}\,\mathrm{m^3\,kg^{-1}}\\ &= \frac{1}{\sigma}10^{-18}\,\mathrm{kg^{-1}\,m\,s^{-1}}\end{aligned}$$

$$\therefore B1 \sim 100 \times C1$$

$$\Longrightarrow \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \approx \frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right)\right]$$

The remaining term on the right of this equation represents the advection of absolute vorticity by the thermal wind. The left hand side, $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega$, is proportional to $-\omega$. When $\omega < 0$ upward vertical motion is implied, and the left hand side is proportional to the vertical velocity. Therefore, upward motion is forced where $\frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right)\right] > 0$, and downward motion is forced where $\frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right)\right] < 0$.

Consider an idealized schematic for a developing synoptic-scale system in the Southern Hemisphere mid-latitudes.



Take note that the 500 hPa contours lead the 1000 hPa contours due to the westward tilt of the system. The result is that the 500 hPa geopotential field lead the isotherm pattern on the figure. The thermal wind, V_T, is parallel to the isotherms, and so the term on the right that represents the advection of absolute vorticity by the thermal wind can be estimated from the change of absolute vorticity along the isotherms.

Keep in mind that we are working here with short-wavelength synoptic-scale systems where relative vorticity advection dominates planetary vorticity advection. Consider the region marked I on the figure of the idealized system. In that region a surface low pressure system is located, and above this surface low at the 500 hPa level the relative vorticity advection is a positive maximum since $\left(\vec{V}_g \cdot \vec{\nabla}\zeta_g\right)_{500\,\text{hPa}} > 0$. This positive advection term is over the surface low pressure center and subsequently contributes to spin-up of the cyclone because the wind is blowing higher positive vorticity into the area of the surface low. However, on the vertical axis of this surface low pressure $\left(\vec{V}_g \cdot \vec{\nabla}\zeta_g\right)_{1000\,\text{hPa}} \approx 0$ because $\left(\vec{\nabla}\zeta_g\right)_{1000\,\text{hPa}} \approx 0$.

Apply the operator $\frac{\partial}{\partial p}$ to the absolute vorticity advection term. We get $\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\left(\zeta_g+f\right)\right) \approx \frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\zeta_g\right)$ for the short-wavelength system considered here.

$$\begin{aligned}\frac{\partial}{\partial p}\left(\vec{V}_g \cdot \vec{\nabla}\zeta_g\right) &= \frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\zeta_g + \vec{V}_g \cdot \frac{\partial}{\partial p}\left(\vec{\nabla}\zeta_g\right) \\ &= \frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\zeta_g \quad \text{since } \zeta_g = \zeta_g(x, y)\end{aligned}$$

We can write

$$\begin{aligned}\frac{\delta \vec{V}_g}{\delta p} \cdot \vec{\nabla}\zeta_g &= \frac{\delta(\vec{V}_g \cdot \vec{\nabla}\zeta_g)}{\delta p} \\ &= \frac{(\vec{V}_g \cdot \vec{\nabla}\zeta_g)_{1000\,\mathrm{hPa}} - (\vec{V}_g \cdot \vec{\nabla}\zeta_g)_{500\,\mathrm{hPa}}}{1000\,\mathrm{hPa} - 500\,\mathrm{hPa}} \\ &\approx \frac{0 - \text{positive value}}{\text{positive value}} < 0 \\ &\implies \frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\zeta_g < 0\end{aligned}$$

For short-wave systems:

$$\begin{aligned}&\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right) < 0 \\ &\therefore \frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right)\right] > 0\end{aligned}$$

In region I where $\frac{f_0}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \cdot \vec{\nabla}\left(\zeta_g + f\right)\right]$ has now been demonstrated to be a positive value, upward motion is forced. Using similar arguments for the region over the surface high pressure system, downward motion is forced. Therefore, upward (downward) motion is forced east (west) of the $500\,\mathrm{hPa}$ trough above the surface low (high) pressure system.

Revisiting the idealized schematic for a developing system, upward motion occurs where relative vorticity increases moving left to right along an isotherm, and downward motion occurs where relative vorticity decreases moving left to right along an isotherm. Notwithstanding the increase in relative vorticity when moving along the isotherm, in the Southern Hemisphere cyclonic storms are associated with negative relative vorticity. Moreover, since $\zeta_g \propto \Phi$ in the Southern Hemisphere the negative vorticity is associated with negative geopotential deviations in region I, which results in the $1000 - 500\,\mathrm{hPa}$ thickness decreasing there leading to a developing trough.

Cold advection occurs behind the cold front, i.e. $\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0$, and warm advection ahead of the warm front, i.e. $\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) < 0$. As a result, the horizontal temperature advection is small above the centre of the surface low in region I. Therefore, in order to cool the atmosphere—as required by the thickness tendency—is by adiabatic (no heat or mass exchange with the environment) cooling through the vertical motion field. As a result, in the presence of differential vorticity advection, the vertical motion maintains a field in which temperature and thickness are proportional (remember that from the thermal wind equation we have seen that $\Phi_1 - \Phi_0 = R\langle T\rangle \ln\left(\frac{p_0}{p_1}\right)$). Because of this proportionality, the vertical motion maintains the temperature field, which is determined by the geopotential field.

The Q-vector

Objective: To better appreciate the essential role of the divergent **a**geostrophic motion in quasi-geostrophic flow.

Here we examine separately the rates of change, following the geostrophic wind, of the vertical shear of the geostrophic wind and of the horizontal temperature gradient.

The approximate horizontal momentum equation:

$$\frac{D_g\vec{V}_g}{Dt} = -f_0\vec{k}\times\vec{V}_a - \beta y\vec{k}\times\vec{V}_g \tag{6.11}$$

Quasi-geostrophic momentum equations:

$$(6.16):\ \frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0 \tag{6.38}$$

$$(6.17):\ \frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0 \tag{6.39}$$

Quasi-geostrophic thermodynamic energy equation

$$\left(\frac{\partial}{\partial t} + \vec{V}_g\cdot\vec{\nabla}\right)T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p} \tag{6.13a}$$

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V}_g\cdot\vec{\nabla}$$

$$\therefore \frac{D_g T}{Dt} - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p} \tag{6.40}$$

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$\text{and } p\frac{\partial u_g}{\partial p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \tag{3.29}$$

On mid-latitude β-plane:

$$f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \quad \text{and} \quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

Vector form:

$$f_0\vec{k}\times\frac{\partial\vec{V}_g}{\partial p} = \frac{R}{p}\vec{\nabla}T$$ (**Bonus homework**)

Obtaining equation for the evolution of the thermal wind components:

$$
\begin{aligned}
& f_0 \frac{\partial}{\partial p}(\textbf{(6.38)}) = f_0 \frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g\right) = 0 \\
& \therefore f_0 \frac{\partial}{\partial p}\left[\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right) u_g\right] - {f_0}^2 \frac{\partial v_a}{\partial p} - f_0 \beta y \frac{\partial v_g}{\partial p} = 0 \\
& \therefore f_0 \left[\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + \frac{\partial}{\partial p}\left(u_g \frac{\partial u_g}{\partial x}\right) + \frac{\partial}{\partial p}\left(v_g \frac{\partial u_g}{\partial y}\right)\right] - {f_0}^2 \frac{\partial v_a}{\partial p} - f_0 \beta y \frac{\partial v_g}{\partial p} = 0 \\
& \therefore f_0 \frac{\partial^2 u_g}{\partial p \partial t} + f_0 \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + f_0 u_g \frac{\partial^2 u_g}{\partial p \partial x} + f_0 \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} + f_0 v_g \frac{\partial^2 u_g}{\partial p \partial y} - {f_0}^2 \frac{\partial v_a}{\partial p} - f_0 \beta y \frac{\partial v_g}{\partial p} = 0 \\
& \therefore f_0 \frac{\partial^2 u_g}{\partial p \partial t} + f_0 u_g \frac{\partial^2 u_g}{\partial p \partial x} + f_0 v_g \frac{\partial^2 u_g}{\partial p \partial y} + f_0 \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + f_0 \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} - {f_0}^2 \frac{\partial v_a}{\partial p} - f_0 \beta y \frac{\partial v_g}{\partial p} = 0 \\
& \therefore f_0 \left[\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right]\frac{\partial u_g}{\partial p} + f_0 \left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] - {f_0}^2 \frac{\partial v_a}{\partial p} - f_0 \beta y \frac{\partial v_g}{\partial p} = 0 \\
\implies & \frac{D_g}{Dt}\left(f_0 \frac{\partial u_g}{\partial p}\right) = -f_0 \left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + {f_0}^2 \frac{\partial v_a}{\partial p} + f_0 \beta y \frac{\partial v_g}{\partial p} \qquad \text{(6.43a)}
\end{aligned}
$$

Similarly:

$$
\frac{D_g}{Dt}\left(f_0 \frac{\partial v_g}{\partial p}\right) = -f_0 \left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] - {f_0}^2 \frac{\partial u_a}{\partial p} - f_0 \beta y \frac{\partial u_g}{\partial p} \qquad \text{(6.43b)}
$$

Bonus homework: Derive (6.43b)

Reminder:

$$
(3.29): \quad f_0 \frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \implies -f_0 \left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]
$$

$$
(3.28): \quad f_0 \frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \implies -f_0 \left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right]
$$

However, the divergence of the geostrophic wind vanishes: $\vec{\nabla} \cdot \vec{V}_g = 0$

$$
\therefore \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \implies \frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y}
$$

$$
\begin{aligned}
\therefore -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] &= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\left(-\frac{\partial v_g}{\partial y}\right) - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] \\
&= \frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] \\
&= -Q_2
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &\equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\vec{i}+\frac{\partial v_g}{\partial y}\vec{j}\right)\cdot\left(\frac{\partial T}{\partial x}\vec{i}+\frac{\partial T}{\partial y}\vec{j}\right) \\
&= -\frac{R}{p}\frac{\partial}{\partial y}\left(u_g\vec{i}+v_g\vec{j}\right)\cdot\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)T \\
&= -\frac{R}{p}\frac{\partial}{\partial y}\vec{V}_g\cdot\vec{\nabla}T \qquad (6.45a)
\end{aligned}
$$

For

$$
\begin{aligned}
\therefore -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right] &= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}+\frac{\partial T}{\partial x}\left(\frac{\partial u_g}{\partial x}\right)\right] \\
&= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] \\
&= Q_1
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &\equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\frac{\partial}{\partial x}\left(u_g\vec{i}+v_g\vec{j}\right)\cdot\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)T \\
&= -\frac{R}{p}\frac{\partial}{\partial x}\vec{V}_g\cdot\vec{\nabla}T \qquad (6.45b)
\end{aligned}
$$

Consider the thermodynamic energy equation

$$
\frac{D_gT}{Dt}-\frac{\sigma p}{R}\omega=\frac{J}{c_p} \qquad (6.40)
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x}(\textbf{(6.40)}) : \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right]T=\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x} \\
&\therefore \frac{\partial^2T}{\partial x\partial t}+\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+u_g\frac{\partial^2T}{\partial x^2}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}+v_g\frac{\partial^2T}{\partial x\partial y}=\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x} \\
&\therefore \frac{\partial^2T}{\partial x\partial t}+u_g\frac{\partial^2T}{\partial x^2}++v_g\frac{\partial^2T}{\partial x\partial y}=-\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x} \\
&\therefore \left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\frac{\partial T}{\partial x}=-\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x}
\end{aligned}
$$

Multiply throughout by $\frac{R}{p}$:

$$
\begin{aligned}
\therefore \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) &= -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial x}+\frac{R}{pc_p}\frac{\partial J}{\partial x} \\
&= -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial x}+\frac{\kappa}{p}\frac{\partial J}{\partial x} \qquad \left[\kappa\equiv\frac{R}{c_p}\right] \qquad (6.46a) \\
\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) &= -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial y}+\frac{\kappa}{p}\frac{\partial J}{\partial y} \qquad (6.46b)
\end{aligned}
$$

Bonus homework: Derive (6.46b). Hint: Start by $\dfrac{\partial}{\partial y}((\mathbf{6.40}))$

Revisiting (6.43a), and remembering

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y}, \quad f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \quad \text{and} \quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

$$\begin{aligned}
\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) &= -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\left(-\frac{\partial v_g}{\partial y}\right) - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= \frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -Q_2 + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p}
\end{aligned} \tag{6.47}$$

and for (6.46b):

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} = Q_2 + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \tag{6.48}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \tag{6.43b}$$

$$\begin{aligned}
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \\
&= Q_1 - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p}
\end{aligned} \tag{6.49}$$

and for (6.46a)

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} = Q_1 + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} \tag{6.50}$$

We set out to examine separately the rates of change $\left(\dfrac{D_g}{Dt}\right)$ of the vertical shear $\left(\dfrac{\partial}{\partial p}\vec{V}_g\right)$ of the geostrophic wind and of the horizontal temperature gradient $(\vec{\nabla}T)$. We subsequently derived two sets of equations describing these relationships. The first set of equations, in Q_2, is:

$$(6.47): \quad \frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -Q_2 + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \qquad \text{(shear)}$$

$$(6.48): \quad \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \qquad \text{(temperature gradient)}$$

(6.48) − (6.47):

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) - Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} - \left[\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) + Q_2 - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p}\right] = 0$$

$$\therefore \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y} - f_0\frac{\partial u_g}{\partial p}\right) - 2Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} = 0$$

Take note that

$$f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \tag{6.41a}$$

$$\therefore \frac{R}{p}\frac{\partial T}{\partial y} - f_0\frac{\partial u_g}{\partial p} = 0$$

$$\therefore -2Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} = 0$$

$$\sigma\frac{\partial \omega}{\partial y} - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = -2Q_2 - \frac{\kappa}{p}\frac{\partial J}{\partial y} \tag{6.51}$$

(6.50) + (6.49):

$$\therefore \frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p} + \frac{R}{p}\frac{\partial T}{\partial x}\right) - Q_1 + {f_0}^2\frac{\partial u_a}{\partial p} + f_0\beta y\frac{\partial u_g}{\partial p} - Q_1 - \sigma\frac{\partial \omega}{\partial x} - \frac{\kappa}{p}\frac{\partial J}{\partial x} = 0$$

Take note that

$$f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \tag{6.41b}$$

$$\therefore f_0\frac{\partial v_g}{\partial p} + \frac{R}{p}\frac{\partial T}{\partial x} = 0$$

$$\therefore -2Q_1 - \frac{\kappa}{p}\frac{\partial J}{\partial x} = -{f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} + \sigma\frac{\partial \omega}{\partial x}$$

$$\sigma\frac{\partial \omega}{\partial x} - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} = -2Q_1 - \frac{\kappa}{p}\frac{\partial J}{\partial x} \tag{6.52}$$

$\frac{\partial}{\partial x}$((**6.52**)):

$$\sigma\frac{\partial^2 \omega}{\partial x^2} - {f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right) - f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right) = -2\frac{\partial Q_1}{\partial x} - \frac{\kappa}{p}\frac{\partial^2 J}{\partial x^2} \tag{A}$$

$\frac{\partial}{\partial y}$((**6.51**)):

$$\sigma\frac{\partial^2 \omega}{\partial y^2} - {f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right) - f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right) - f_0\beta\frac{\partial v_g}{\partial p} = -2\frac{\partial Q_2}{\partial y} - \frac{\kappa}{p}\frac{\partial^2 J}{\partial y^2} \tag{B}$$

(A)+(B):

$$\sigma\frac{\partial^2\omega}{\partial x^2}+\sigma\frac{\partial^2\omega}{\partial y^2}-{f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right)-{f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right)-f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right)$$
$$-f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right)-f_0\beta\frac{\partial v_g}{\partial p}=-2\frac{\partial Q_1}{\partial x}-2\frac{\partial Q_2}{\partial y}-\frac{\kappa}{p}\frac{\partial^2 J}{\partial x^2}-\frac{\kappa}{p}\frac{\partial^2 J}{\partial y^2}$$
$$\therefore\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right)-{f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right)-f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right)$$
$$-f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right)-f_0\beta\frac{\partial v_g}{\partial p}=-2\vec{\nabla}\cdot\vec{Q}-\frac{\kappa}{p}\nabla^2 J$$

We have:

1) The divergence of the geostrophic wind vanishes:

$$\vec{\nabla}\cdot\vec{V}_g=0$$
$$\therefore\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}=0$$
$$\therefore\frac{\partial u_g}{\partial x}=-\frac{\partial v_g}{\partial y}$$

2) (6.12): $\dfrac{\partial u_a}{\partial x}+\dfrac{\partial v_a}{\partial y}=-\dfrac{\partial\omega}{\partial p}$

$$\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial p}\left(\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right)-f_0\beta y\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}\right)-f_0\beta\frac{\partial v_g}{\partial p}$$
$$=-2\vec{\nabla}\cdot\vec{Q}-\frac{\kappa}{p}\nabla^2 J$$
$$\therefore\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial p}\left(-\frac{\partial\omega}{\partial p}\right)=-2\vec{\nabla}\cdot\vec{Q}+f_0\beta\frac{\partial v_g}{\partial p}-\frac{\kappa}{p}\nabla^2 J$$
$$\therefore\sigma\nabla^2\omega+{f_0}^2\frac{\partial^2\omega}{\partial p^2}=-2\vec{\nabla}\cdot\vec{Q}+f_0\beta\frac{\partial v_g}{\partial p}-\frac{\kappa}{p}\nabla^2 J \tag{6.53}$$

$\implies$ the Q-vector form of the omega equation.

From (6.45a,b):

$$\vec{Q}=(Q_1,Q_2)$$
$$=\left(-\frac{R}{p}\frac{\partial}{\partial x}\vec{V}_g\cdot\vec{\nabla}T,-\frac{R}{p}\frac{\partial}{\partial y}\vec{V}_g\cdot\vec{\nabla}T\right) \tag{6.54}$$

Outside regions of active precipitation, diabatic heating is due primarily to net radiative heating, which is weak in the troposphere. Therefore, the Laplacian of the diabatic heating can be neglected. Also, the term related to the beta (β) effect is generally small for synoptic scale motion, and is subsequently also neglected. The resulting Q vector form of the omega equation is

$$\sigma\nabla^2\omega+{f_0}^2\frac{\partial^2\omega}{\partial p^2}=-2\vec{\nabla}\cdot\vec{Q}$$

Next we will discuss $\vec{Q}$ as a forcing function of the omega equation.

Previously it was demonstrated that $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$

Multiplying throughout with σ ($\sigma > 0$): $\left(\sigma\nabla^2 + {f_0}^2\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$

So when $\left({f_0}^2\frac{\partial^2}{\partial p^2} + \sigma\nabla^2\right)\omega > 0$, we have ascending (upward) motion ($\omega < 0$)

$$\therefore -2\vec{\nabla}\cdot\vec{Q} > 0$$
$$\therefore \vec{\nabla}\cdot\vec{Q} < 0$$

$\therefore$ Negative divergence of $\vec{Q}$, i.e. convergence of $\vec{Q}$, leads to ascending motion.

Similarly, when $\left({f_0}^2\frac{\partial^2}{\partial p^2} + \sigma\nabla^2\right)\omega < 0$, we have descending (downward) motion ($\omega > 0$)

$$\therefore -2\vec{\nabla}\cdot\vec{Q} < 0$$
$$\therefore \vec{\nabla}\cdot\vec{Q} > 0$$

$\therefore$ Divergence of $\vec{Q}$ leads to descending motion.

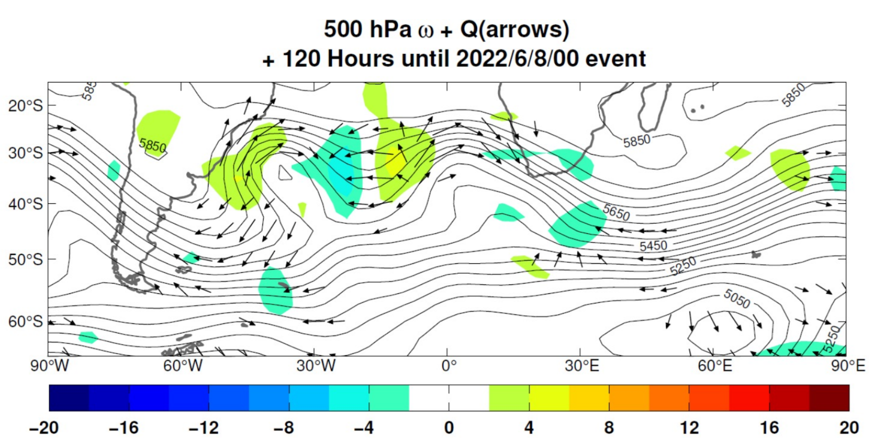


Figure 1q: Omega

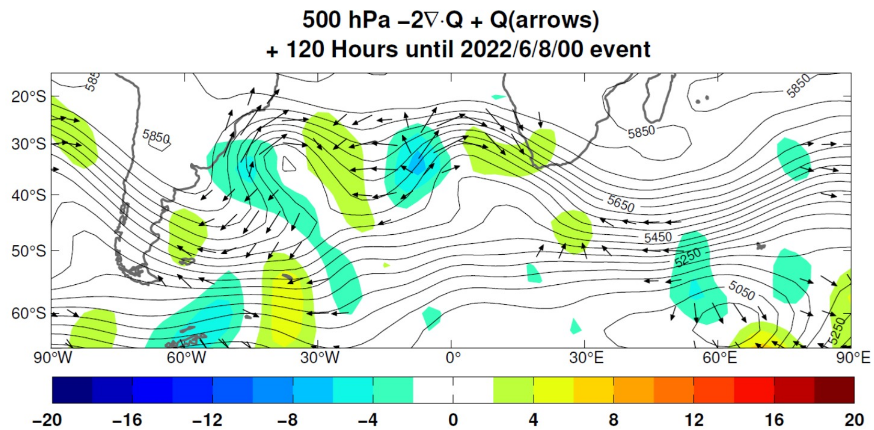
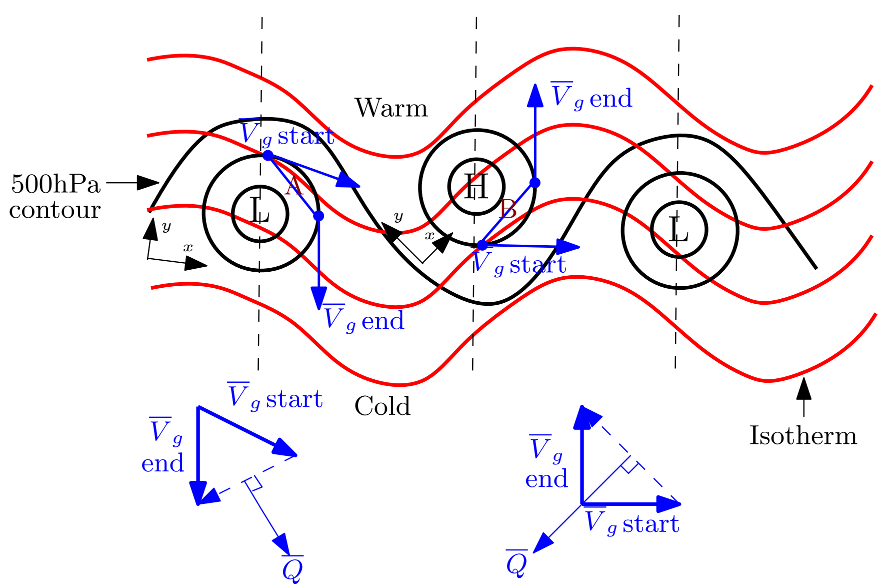


Figure 2q: Divergence

Figures 1q and 2q respectively show the omega (negative for upward motion) and convergence (negative divergence) of the Q-vector fields as observed on 2022-06-08. The units are 10^{-18} m/kg/s for the divergence and 10^{-14} m/kg/s for the Q-vectors. Shown in the diagrams are the values multiplied by 10^{18} (divergence) and 10^{14} (vectors). Converging (diverging) Q-vectors show where upward (ascending) motion is found and is for the most part in agreement with the omega fields.



Consider an idealized developing synoptic-scale system in the Southern Hemisphere mid-latitudes at the 500hPa level. The Q-vector direction and magnitude can be estimated by referring the motion to a Cartesian coordinate system. In this coordinate system, the x-axis is parallel to the local isotherm and the y-axis is perpendicular to the isotherm. Since warm air is to the left of an observer moving along an isotherm, temperature increases in the positive y-direction in the Southern Hemisphere. In this configuration, the

Q-vector may be simplified:

$$\begin{aligned}
Q_1 &= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} \quad \text{since } \frac{\partial T}{\partial x} = 0 \text{ (x-axis parallel to isotherm)} \\
Q_2 &= -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} \\
&= \frac{R}{p}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y} \quad \text{since } \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \\
\implies \vec{Q} &= -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\vec{i} - \frac{\partial u_g}{\partial x}\vec{j}\right) \\
\text{consider } -\vec{k} \times \left(\frac{\partial u_g}{\partial x}\vec{i} + \frac{\partial v_g}{\partial x}\vec{j}\right) &= -\vec{k} \times \frac{\partial \vec{V}_g}{\partial x} \\
&= -\frac{\partial u_g}{\partial x}\vec{j} + \frac{\partial v_g}{\partial x}\vec{i} \\
\therefore \vec{Q} &= -\frac{R}{p}\frac{\partial T}{\partial y}\left(-\vec{k} \times \frac{\partial \vec{V}_g}{\partial x}\right) \\
&= \frac{R}{p}\frac{\partial T}{\partial y}\left(\vec{k} \times \frac{\partial \vec{V}_g}{\partial x}\right)
\end{aligned}$$

The Q-vector can be obtained by considering the vectorial change of $\vec{V}_g$ along the isotherm. Consider two cases, A and B, of an observer moving along an isotherm. For each case, draw an arrow describing the geostrophic wind vector observed at the start of each movement. Draw a second arrow showing the geostrophic wind vector at the end of each movement. Next, draw the vector difference from the head of the start vector to the head of the end vector. The Q-vector direction points $90°$ to the left (anti-clockwise) from the geostrophic difference vector in the Southern Hemisphere as dictated by the reduced Q-vector equation above. The resulting vector, multiplied by $\partial T/\partial y$, provides its magnitude.

Near the $500\mathrm{hPa}$ low, the geostrophic wind change vector produces a Q-vector parallel to the thermal wind, while near the high the Q-vector is anti-parallel to the thermal wind. The two Q-vectors thus converge in the area between the trough and ridge lines where we have already shown upward motion to occur.

Ageostrophic flow

The characteristic horizontal scale of the geostrophic wind in the mid-latitude troposphere is about 10 to $20\,\mathrm{m\,s^{-1}}$, while the scale of the ageostrophic wind is an order of magnitude smaller, often only $1-2\,\mathrm{m\,s^{-1}}$. Although the ageostrophic flow is only a small component of the wind field, the upward motion, omega (ω), is determined only by its ageostrophic part. Here we will further demonstrate the significance of the ageostrophic wind components.

Consider the following thermal wind relationship

$$f_0 \frac{\partial u_g}{\partial p} = \frac{R}{p} \frac{\partial T}{\partial y}$$

Then the evolution of the thermal wind components leads to

$$\frac{D_g}{Dt} \left(f_0 \frac{\partial u_g}{\partial p} \right) = -Q_2 + {f_0}^2 \frac{\partial v_a}{\partial p} + f_0 \beta y \frac{\partial v_g}{\partial p} \tag{1}$$

$$\frac{D_g}{Dt} \left(\frac{R}{p} \frac{\partial T}{\partial y} \right) = Q_2 + \sigma \frac{\partial \omega}{\partial y} + \frac{\kappa}{p} \frac{\partial J}{\partial y} \tag{2}$$

Assume that diabatic heating is small enough to disregard and consider the flow to be purely geostrophic, i.e. $\vec{V}_a = \vec{0}$, and $\omega = 0$ because ω is determined only by the ageostrophic part of the wind field:

$$-\frac{\partial \omega}{\partial p} = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

Equations (1) and (2) are reduced to

$$\frac{D_g}{Dt} \left(f_0 \frac{\partial u_g}{\partial p} \right) = -Q_2 + f_0 \beta y \frac{\partial v_g}{\partial p} \tag{3}$$

$$\frac{D_g}{Dt} \left(\frac{R}{p} \frac{\partial T}{\partial y} \right) = Q_2 \tag{4}$$

Scale analysis is subsequently performed in order to estimate the magnitudes of the various terms of equation (3). First, consider the left-hand side of equation (3)

$$\frac{D_g}{Dt} \left(f_0 \frac{\partial u_g}{\partial p} \right) = \frac{\partial}{\partial t} \left(f_0 \frac{\partial u_g}{\partial p} \right) + u_g \frac{\partial}{\partial x} \left(f_0 \frac{\partial u_g}{\partial p} \right) + v_g \frac{\partial}{\partial y} \left(f_0 \frac{\partial u_g}{\partial p} \right)$$

$$\frac{\partial}{\partial t} \left(f_0 \frac{\partial u_g}{\partial p} \right) \sim \frac{1}{L/U} f_0 \frac{U}{\delta p} = \frac{f_0 U^2}{L \delta p}$$

$$u_g \frac{\partial}{\partial x} \left(f_0 \frac{\partial u_g}{\partial p} \right) \text{ and } v_g \frac{\partial}{\partial y} \left(f_0 \frac{\partial u_g}{\partial p} \right) \sim U \frac{1}{L} f_0 \frac{U}{\delta p} = \frac{f_0 U^2}{L \delta p}$$

Consider the right-hand side of equation (3)

$Q_2 = -\frac{R}{p} \left[\frac{\partial u_g}{\partial y} \frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y} \frac{\partial T}{\partial y} \right]$, but consider (3.28) and (3.29)

$$p \frac{\partial v_g}{\partial p} = -\frac{R}{f} \frac{\partial T}{\partial x} \implies \frac{\partial T}{\partial x} = -\frac{fp}{R} \frac{\partial v_g}{\partial p},$$

$$p \frac{\partial u_g}{\partial p} = \frac{R}{f} \frac{\partial T}{\partial y} \implies \frac{\partial T}{\partial y} = \frac{fp}{R} \frac{\partial u_g}{\partial p}$$

$$\therefore Q_2 = -\frac{R}{p}\frac{fp}{R}\left[-\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} + \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right]$$
$$= f\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right]$$
$$= (f_0 + \beta y)\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right]$$

Assuming that the meridional displacement, y, is at most equal to the length scale of $10^6\,\mathrm{m}$, then

$$\beta y \sim 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1} \times (10^6\,\mathrm{m})$$
$$= 10^{-5}\,\mathrm{s}^{-1} \ll f_0$$

$$\therefore Q_2 = f_0\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right]$$
$$\sim \frac{f_0 U^2}{L\delta p}$$

Scale analysis has therefore shown that the order of magnitude of the left-hand side of the equation and Q_2 is the same, which is

$$\frac{f_0 U^2}{L\delta p} \sim \frac{10^{-4}\,\mathrm{s}^{-1} \times (10\,\mathrm{m\,s}^{-1})^2}{(10^6\,\mathrm{m}) \times 1000\,\mathrm{kg\,m}^{-1}\,\mathrm{s}^{-2}}$$
$$= 10^{-11}\,\mathrm{kg}^{-1}\,\mathrm{m}^2\,\mathrm{s}^{-1}$$

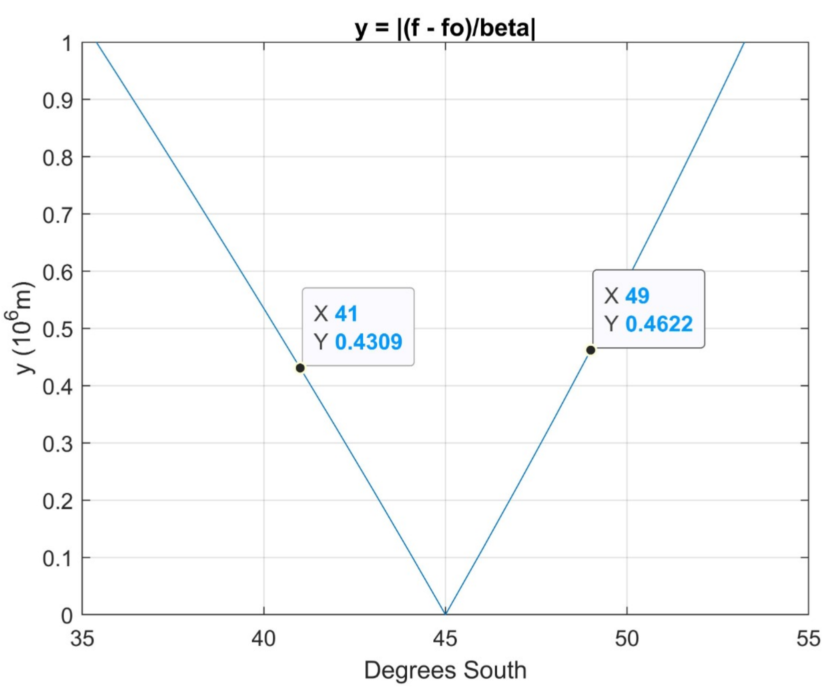


Figure 16: Meridional displacement in absolute terms in the Southern Hemisphere.

Since the meridional displacement $y = \beta^{-1}(f - f_0)$, the figure above shows that meridional displacements between about $41°$S and $49°$S, y can be approximated by $10^5\,\mathrm{m}$. As a result, scale analysis of the β term of equation (3) results in

$$f_0\beta y\frac{\partial v_g}{\partial p} \sim \frac{10^{-4}\,\mathrm{s}^{-1} \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1} \times 10^5\,\mathrm{m} \times 10\,\mathrm{m}\,\mathrm{s}^{-1}}{1000\,\mathrm{kg}\,\mathrm{m}^{-1}\,\mathrm{s}^{-2}}$$
$$= 10^{-12}\,\mathrm{kg}^{-1}\,\mathrm{m}^2\,\mathrm{s}^{-1}$$

The β term is therefore an order of magnitude smaller than the rest of the terms and is subsequently disregarded. This result leads to

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -Q_2$$
$$\therefore Q_2 = -\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right)$$
$$= -\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) \text{, because of the the thermal wind relationship.}$$

However, the reduced form of equation (4) is

$$Q_2 = \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right),$$

which contradicts the scaled result for Q_2. The implication is that in order to address this contradiction is for either the vertical shear $\left(\frac{\partial u_g}{\partial p}\right)$ or the temperature gradient $\left(\frac{\partial T}{\partial y}\right)$ to vanish or be a constant. We can therefore not ignore the ageostrophic wind terms, and so the ageostrophic circulation is required to keep the flow in approximate thermal wind balance.

The role of ageostrophic circulation in vertical motion is implied through the determination of the omega (ω) motion field, since ω is determined only by the ageostrophic part of the wind field

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = -\frac{\partial \omega}{\partial p}$$
$$\vec{\nabla} \cdot \vec{V}_a = -\frac{\partial \omega}{\partial p}$$

However, the total ageostrophic flow field cannot be determined by the divergence alone, because

$$\frac{D_g\vec{V}_g}{Dt} = -f_0\vec{k} \times \vec{V}_a - \beta y\vec{k} \times \vec{V}_g$$

By again neglecting the β effect for simplicity

$$\frac{D_g\vec{V}_g}{Dt} = -f_0\vec{k} \times \vec{V}_a$$
$$\vec{k} \times \vec{k} \times \vec{V}_a = -\frac{1}{f_0}\vec{k} \times \frac{D_g\vec{V}_g}{Dt}$$

The left-hand side is

$$\begin{aligned}
\vec{k}\times\vec{k}\times\vec{V}_a &= \vec{k}\times\vec{k}\times(V_{a1}\vec{i}+V_{a2}\vec{j})\\
&= \vec{k}\times\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & 1\\ V_{a1} & V_{a2} & 0\end{vmatrix}\\
&= \vec{k}\times(\vec{i}(-V_{a2})-\vec{j}(-V_{a1}))\\
&= -V_{a2}\vec{j}-(-\vec{i})(-V_{a1})\\
&= -(V_{a1}\vec{i}+V_{a2}\vec{j})\\
&= -\vec{V}_a\\
\therefore -\vec{V}_a &= -\frac{1}{f_0}\vec{k}\times\frac{D_g\vec{V}_g}{Dt}\\
\therefore \vec{V}_a &= \frac{1}{f_0}\vec{k}\times\left(\frac{\partial\vec{V}_g}{\partial t}+\vec{V}_g\cdot\vec{\nabla}\vec{V}_g\right)\\
&= \frac{1}{f_0}\vec{k}\times\frac{\partial\vec{V}_g}{\partial t}+\frac{1}{f_0}\vec{k}\times\vec{V}_g\cdot\vec{\nabla}\vec{V}_g
\end{aligned}$$

The ageostrophic wind forcing therefore consists of two parts. The first term on the right represents the isallobaric[1] wind, and the second term is called the advective part of the ageostrophic wind.

Consider the isallobaric term

$$\begin{aligned}
\frac{1}{f_0}\vec{k}\times\frac{\partial}{\partial t}\vec{V}_g &= \frac{1}{f_0}\vec{k}\times\frac{\partial}{\partial t}\left(\frac{1}{f_0}\vec{k}\times\vec{\nabla}\Phi\right)\\
&= \frac{1}{{f_0}^2}\vec{k}\times\vec{k}\times\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right)\\
&= \frac{1}{{f_0}^2}\vec{k}\times\vec{k}\times\vec{\nabla}\chi\\
&= \frac{1}{{f_0}^2}\vec{k}\times\vec{k}\times\left(\frac{\partial\chi}{\partial x}\vec{i}+\frac{\partial\chi}{\partial y}\vec{j}\right)\\
&= \frac{1}{{f_0}^2}\vec{k}\times\left(\frac{\partial\chi}{\partial x}\vec{j}-\frac{\partial\chi}{\partial y}\vec{i}\right)\\
&= \frac{1}{{f_0}^2}\left(-\frac{\partial\chi}{\partial x}\vec{i}-\frac{\partial\chi}{\partial y}\vec{j}\right)\\
&= -\frac{1}{{f_0}^2}\vec{\nabla}\chi
\end{aligned}$$

Therefore, the isallobaric wind is proportional to the gradient of the geostrophic tendency. Since ${f_0}^2$ is involved, there is no change of sign in crossing the equator. This isallobaric wind blows towards falling geopotential in both hemispheres.

Next consider the term that is the advective part of the ageostrophic wind. At the synoptic scale, baroclinic waves grow in the mid-latitudes due to baroclinic instability (arising from vertical shear of the mean

[1] of equal or constant pressure change.

flow and thermal wind). When such waves are part of the jet stream, the advective term is dominated by zonal advection, since jet streams are quasi-horizontal with maximum winds embedded in the mid-latitudes westerlies. Let $\vec{u}$ denote the mean zonal flow, then

$$\begin{aligned}
\vec{V}_g \cdot \vec{\nabla}\vec{V}_g &\simeq (\vec{u}\vec{i} + 0\vec{j}) \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\vec{V}_g \\
&= \vec{u}\frac{\partial}{\partial x}\vec{V}_g \\
\therefore \frac{1}{f_0}\vec{k} \times \left(\vec{u}\frac{\partial}{\partial x}\vec{V}_g\right) &= \frac{1}{f_0}\vec{u}\frac{\partial}{\partial x}(\vec{k} \times \vec{V}_g) \\
&= \frac{1}{f_0}\vec{u}\frac{\partial}{\partial x}\left(\vec{k} \times \left(\frac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi\right)\right) \\
&= \frac{1}{{f_0}^2}\vec{u}\frac{\partial}{\partial x}\left(\vec{k} \times \left(\vec{k} \times \left(\frac{\partial\Phi}{\partial x}\vec{i} + \frac{\partial\Phi}{\partial y}\vec{j}\right)\right)\right) \\
&= \frac{1}{{f_0}^2}\vec{u}\frac{\partial}{\partial x}\left(-\frac{\partial\Phi}{\partial x}\vec{i} - \frac{\partial\Phi}{\partial y}\vec{j}\right) \\
&= -\frac{1}{{f_0}^2}\vec{u}\frac{\partial}{\partial x}(\vec{\nabla}\Phi)
\end{aligned}$$

$\Longrightarrow$ The ageostrophic wind

$$\vec{V}_a = -\frac{1}{{f_0}^2}\left[\vec{\nabla}\chi + \vec{u}\frac{\partial}{\partial x}(\vec{\nabla}\Phi)\right]$$

Next, we will perform a scale analysis on this ageostrophic wind equation. First, do the scale analysis of the isallobaric wind

$$\begin{aligned}
\vec{V}_{\text{isall}} &= -\frac{1}{{f_0}^2}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) \\
&\sim -\frac{1}{f^2}\frac{1}{L}\left(\frac{L}{U}\right)^{-1}\delta\Phi \\
&= -\frac{1}{f^2}\frac{U}{L^2}\left(-\frac{1}{\rho}\delta p\right) \\
&= \frac{U\delta p}{f^2L^2\rho} \\
&\sim \frac{10\,\mathrm{m\,s^{-1}} \times \left(10 \times 10^2\,\mathrm{kg\,m\,s^{-2}\,m^{-2}}\right)}{10^{-8}\,\mathrm{s^{-2}} \times (10^6\,\mathrm{m})^2 \times 1\,\mathrm{kg\,m^{-3}}} \\
&= \frac{10^4}{10^4}\frac{\mathrm{s^{-3}}}{\mathrm{s^{-2}\,m^{-1}}} = 1\,\mathrm{m\,s^{-1}}
\end{aligned}$$

Next, scale the term that is the advective part of the ageostrophic wind

$$
\begin{aligned}
& -\frac{1}{{f_0}^2}\vec{u}\frac{\partial}{\partial x}\vec{\nabla}\Phi \\
& \sim -\frac{1}{f^2}U\frac{1}{L}\frac{1}{L}\delta\Phi \\
& = -\frac{1}{f^2}\frac{U}{L^2}\left(-\frac{1}{\rho}\delta p\right) \\
& = \frac{U\delta p}{f^2L^2\rho} \text{ , which is the same as was found for the isallobaric wind.}
\end{aligned}
$$

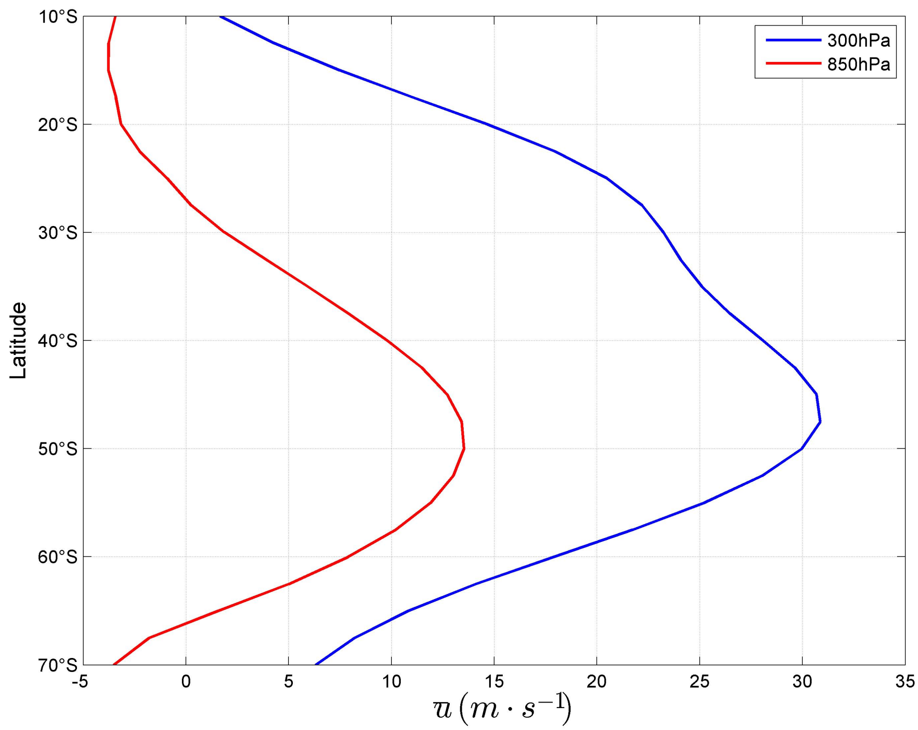


Figure 17: Mean zonal flow distribution at the two pressure levels indicated.

The scale analysis done here shows that both the isallobaric wind and the advective part are about $1\,\mathrm{m\,s^{-1}}$, given both the typical horizontal wind speed and the mean zonal flow to be $10\,\mathrm{m\,s^{-1}}$. However, profiles of the time-mean zonal geostrophic wind, averaged over longitudes, show that for two isobaric levels, one at $850\,\mathrm{hPa}$ and the other at $300\,\mathrm{hPa}$, that there is a strong jetstream at $300\,\mathrm{hPa}$ in the mid-latitudes.

Zonal mean winds of the $300\,\mathrm{hPa}$ jetstream are typically of the order of 30 to $40\,\mathrm{m\,s^{-1}}$ in the mid-latitudes, while at $850\,\mathrm{hPa}$ the zonal maximum wind is closer to the $10\,\mathrm{m\,s^{-1}}$ value used in the scale analysis. Therefore, at the $300\,\mathrm{hPa}$ jetstream the advective contribution to the ageostrophic wind dominates over the isallobaric contribution. At both high and low latitudes, that is on the edges or flanks of mid-latitude baroclinic systems, zonal wind at the $300\,\mathrm{hPa}$ level are of similar strength and weak. Thus, at the flanks the two contributions to the ageostrophic flow are similarly small so that the resulting ageostrophic wind is small. The net effect is that at $300\,\mathrm{hPa}$ the ageostrophic motion is primarily zonal.

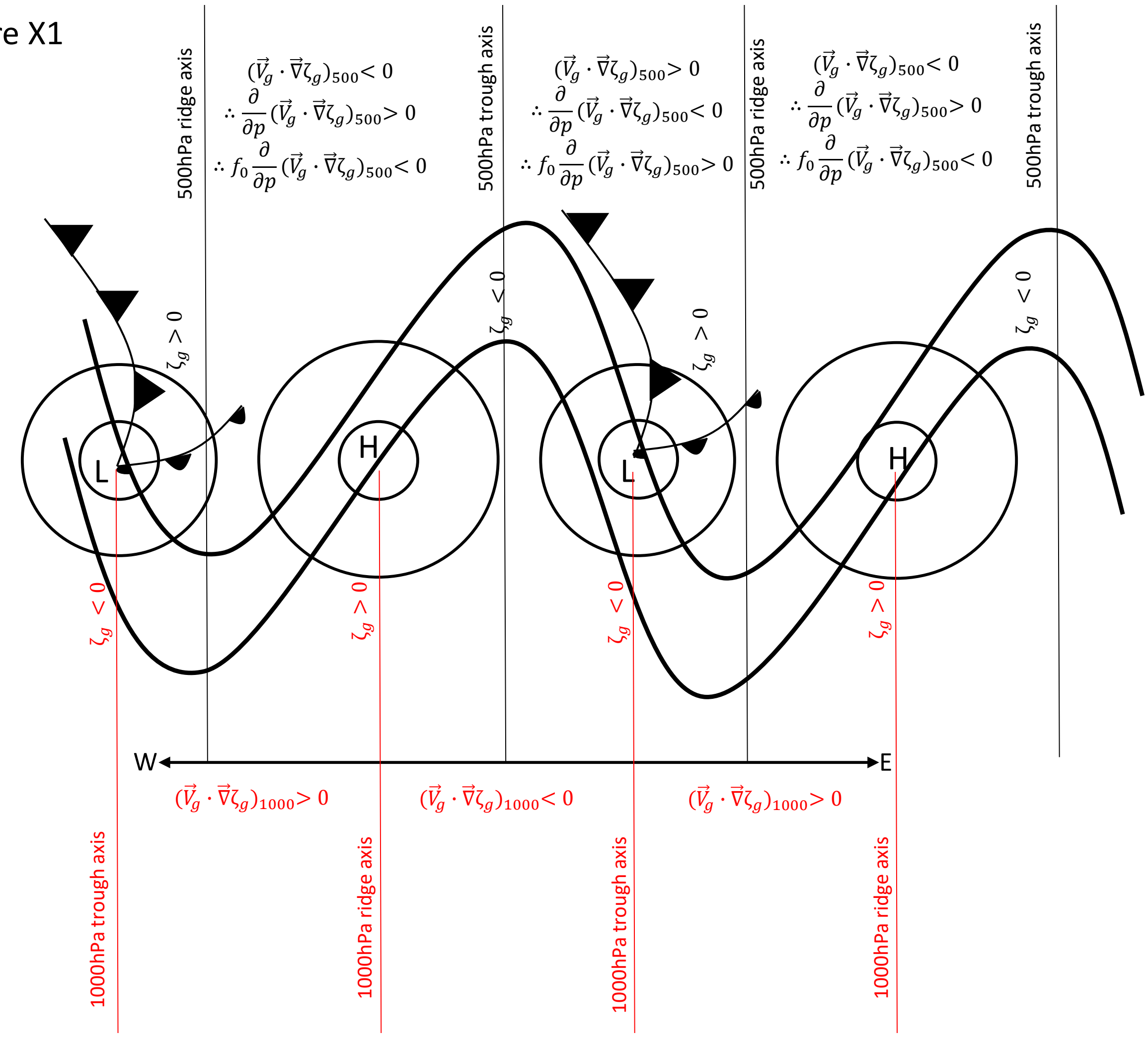
At the $850\,\mathrm{hPa}$ level, owing to the relatively weak zonal wind in the mid-latitudes, the advective part does not dominate as it does at the $300\,\mathrm{hPa}$ level. Moreover, owing to the weak wind at $850\,\mathrm{hPa}$ at the flanks, the advective contribution is nearly zero there. On the other hand, since the isallobaric wind is always directed down the gradient of pressure tendency, it will cause a meridional component in both the northern and southern flanks of the baroclinic wave, and a zonal component along the central axis of the wave. However, along the zonal axis the isallobaric contribution may balance that of the advective contribution, with the net result that the ageostrophic motion will only have a meridional component in the northern and southern flanks at $850\,\mathrm{hPa}$.

Vertical and horizontal motions in a developing short-wave baroclinic system in the Southern Hemisphere – a summary

This chapter on the quasi-geostrophic theory started off by presenting a classic theory on the motion of mid-latitude developing baroclinic systems in the Southern Hemisphere. In this theory the atmospheric level of non-divergence (transition from positive to negative divergence, and vice versa) was introduced and it was concluded that if this level is low enough in altitude that the system will move eastward. The eastward movement of these systems fits into the theory of short-wave developing baroclinic systems discussed in this chapter. The $500\,\mathrm{hPa}$ level is often assumed to be the level of non-divergence, and is about halfway through the vertical depth of the mass of the atmosphere. Next, through the use of highly idealistic circulation fields at the $500\,\mathrm{hPa}$ level the relative importance of the advections of relative and planetary vorticity was investigated – relative vorticity advection dominates planetary vorticity advection for short-wave systems. Moreover, for fast-moving extratropical short-wave weather systems that are not rapidly amplifying or decaying the local rate of change of geostrophic vorticity is represented only by the advection of geostrophic vorticity. However, in the presence of developing systems this rate is also a function of the divergence effect, which forms part of the ageostrophic flow.

Consider Figure X1 that shows a mean sea-level (i.e., $1000\,\mathrm{hPa}$) pressure pattern represented by fine lines and $500\,\mathrm{hPa}$ pattern by thick lines for the Southern Hemisphere. First we will consider vorticity advection for short-wave systems at both the $1000\,\mathrm{hPa}$ and $500\,\mathrm{hPa}$ levels, by neglecting the effect of planetary vorticity advection. The vertical lines on the figure respectively represent trough and ridge axes at $500\,\mathrm{hPa}$ as well as at the surface. At the surface by the centres of both the low and the high pressure systems, the geostrophic vorticity advection is close to zero. However, at the $500\,\mathrm{hPa}$ level above the surface low (high) pressure system the advection of geostrophic vorticity is higher (lower) and positive (negative). Vorticity advection is usually higher in absolute terms at $500\,\mathrm{hPa}$ than at the surface because the wind speeds tend to increase with height, therefore $500\,\mathrm{hPa}$ winds near a trough will often be stronger than low-level winds. Accordingly, there is a positive change in the vertical of the geostrophic vorticity advection, referred to as differential vorticity advection, which is a negative value above the surface low owing to the decrease in pressure with increasing height above the surface. However, in the Southern Hemisphere where the Coriolis parameter is negative, multiplying the differential vorticity advection term with f_0 leads to a positive differential vorticity advection term. Take note that rising air is implied by an increase with height of cyclonic relative vorticity advection, which is the case above the surface low.

Figure X1



The positive advection of relative vorticity at the $500\,\mathrm{hPa}$ level above the surface low pressure system has implications for the horizontal displacement of mid-latitude disturbances. Consider Figure X2 as well as the geopotential tendency equation. This equation consists of two terms on the right respectively representing vorticity advection (the dominant forcing term in the upper troposphere) and thickness or temperature advection (largest in magnitude in the lower troposphere). Due to the advection of relative vorticity at $500\,\mathrm{hPa}$ above the surface low, geopotential heights are decreasing, while geopotential heights are increasing at $500\,\mathrm{hPa}$ above the surface low. Vorticity advection at the $500\,\mathrm{hPa}$ level thus acts to propagate the disturbance horizontally to spread it vertically. Regarding the temperature advection term of the tendency equation, below the $500\,\mathrm{hPa}$ trough cold advection in association with the cold front occurs, while warm advection in association with the warm front occurs below the $500\,\mathrm{hPa}$ ridge. The effect of cold advection below the $500\,\mathrm{hPa}$ trough is to deepen the trough in the upper troposphere, while the effect of the warm advection below the $500\,\mathrm{hPa}$ ridge is to build the ridge in the upper troposphere. What is the source of this advection? Recalling the figure of a developing synoptic-scale system in the section that discussed the Sutcliffe form of the omega equation, the $500\,\mathrm{hPa}$ contours lead the $1000\,\mathrm{hPa}$ contours due to the westward tilt of the developing system. During this development the result is that the $500\,\mathrm{hPa}$ geopotential field leads the isotherm pattern. While the angle between the geopotential height contours and the thickness contours increases, an increase in the horizontal temperature advection is the result. However, as the system is allowed to further develop, the surface low pressure contours, the $500\,\mathrm{hPa}$ contours and the thickness contours come into alignment with each other. This later stage of development results in the weakening of the horizontal temperature advection and marks the end of the intensification phase in the lifecycle of the short-wave baroclinic system.

The omega equation was used to determine where upward and downward motion in a developing system may occur. Figure X3 shows the results from analyzing three versions of this equation, namely its traditional form that consists of two terms on the right respectively representing differential vorticity advection and thickness (temperature) advection, the reduced Sutcliffe form that only has one term on the right representing the advection of vorticity by the thermal wind, and the Q-vector form. The differential vorticity advection terms have shown that upward (downward) motion in the developing system occurs over the surface low (high) pressure systems, and that the thickness advection forces upward (downward) motion at the $500\,\mathrm{hPa}$ ridge (trough) axis ahead (behind) the warm (cold) front. Although the interpretations of these two physical processes have apparent advantages as demonstrated here, in practice there is often a significant amount of cancellation between them. For this reason an alternative, albeit an approximate form of the omega equation, is often applied in synoptic analyses – hence the Sutcliffe version of the equation. This version showed that upward (downward) motion is forced east (west) of the $500\,\mathrm{hPa}$ trough above the surface low (high) pressure system. This finding is in agreement with the interpretation of the Q-vector form of the omega equation as shown above.

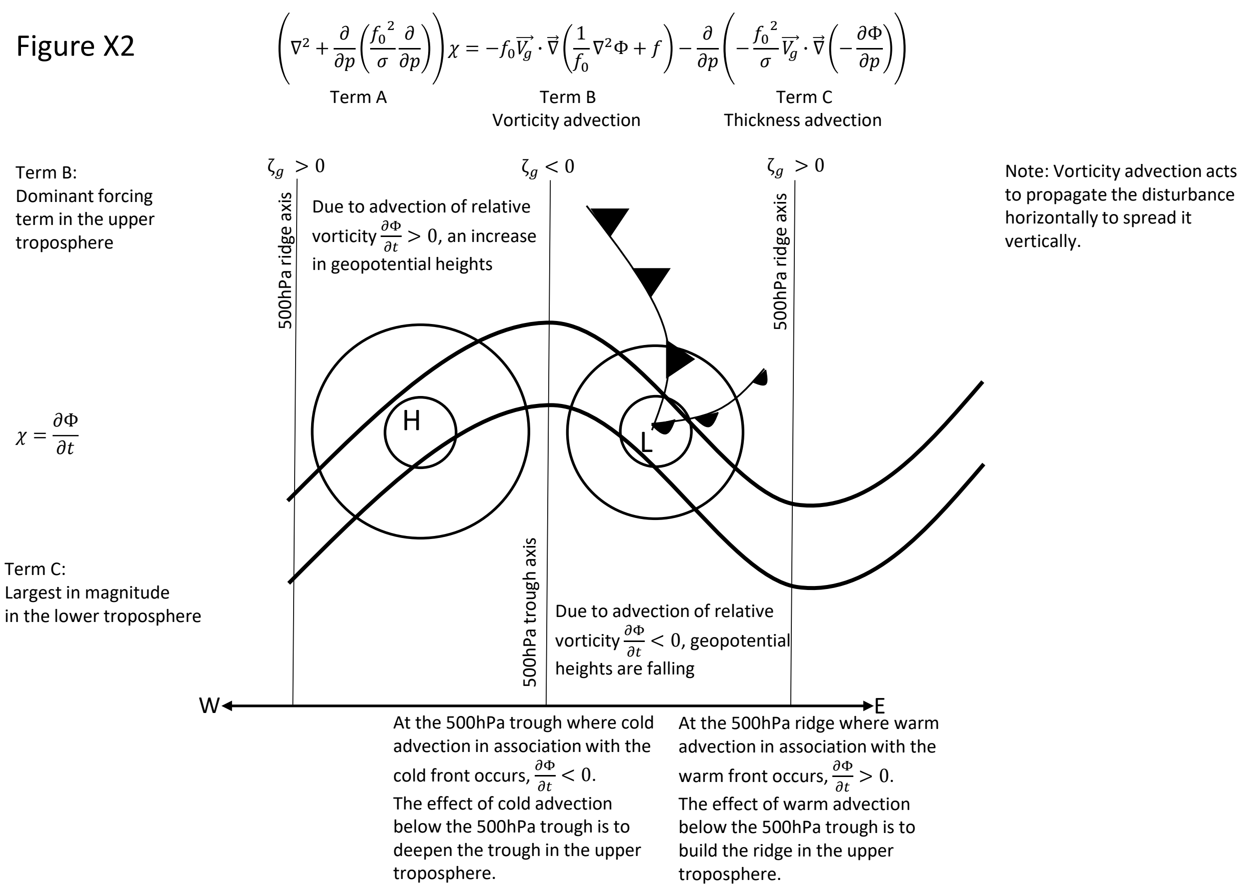
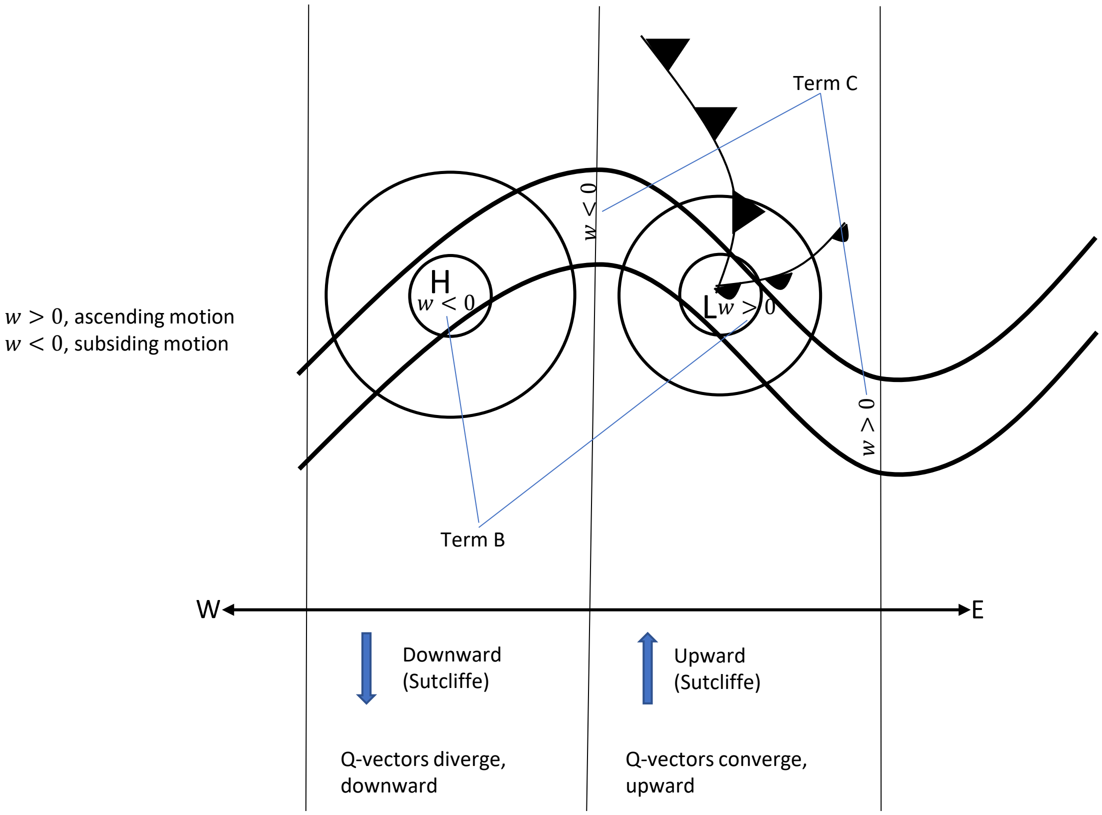


Figure X3

$$\left(\nabla^2 + \frac{f_0^{\,2}}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{V_g}\cdot\vec{\nabla}(\zeta_g+f)\right) + \frac{1}{\sigma}\nabla^2\left(\vec{V_g}\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right) \approx \frac{f_0}{\sigma}\left(\frac{\partial\vec{V_g}}{\partial p}\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right)$$

Term A — Term B: Differential vorticity advection — Term C: Thickness advection — Sutcliffe: Advection of vorticity by the thermal wind



To summarize, temperature advection forces the strengthening of mid-tropospheric troughs and ridges, the advection of relative vorticity acts to propagate the developing system horizontally, while differential relative vorticity advection forces rising (sinking) motion over surface low (high) pressure systems, as is the advection of vorticity by the thermal wind.

Linear perturbation theory

We wish to gain physical insight of atmospheric waves by employing simplified models. Here we introduce the perturbation method to examine atmospheric waves and then to gain further insight into the development of synoptic wave disturbances in terms of their origin and propagation.

Variables are divided into two parts consisting of a basic state portion and a perturbation portion. The basic state portion is independent of time and longitude. Recall the total derivative of the x-component of the eastward velocity, u:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$u\dfrac{\partial u}{\partial x}$ is the zonal part of advective acceleration and is non-linear. We want to expand this acceleration by expanding the complete velocity field, $u(x,t)$, into a time and longitude-average zonal velocity, $\overline{u}$, and u', the deviation from that average:

$$u(x,t) = \overline{u} + u'(x,t).$$

The acceleration term then becomes:

$$\begin{aligned} u\frac{\partial u}{\partial x} &= (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') \\ &= \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial u'}{\partial x} \\ &= \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} \quad \text{since } \overline{u} \text{ is independent of longitude.} \end{aligned}$$

However, $\left|\dfrac{u'}{\overline{u}}\right| \ll 1$, therefore $|\overline{u}| \gg |u'|$, which leads to $\left|\overline{u}\dfrac{\partial u'}{\partial x}\right| \gg \left|u'\dfrac{\partial u'}{\partial x}\right|$. Therefore, terms that involve products of the perturbations may be neglected.

In the case of purely zonal flows:

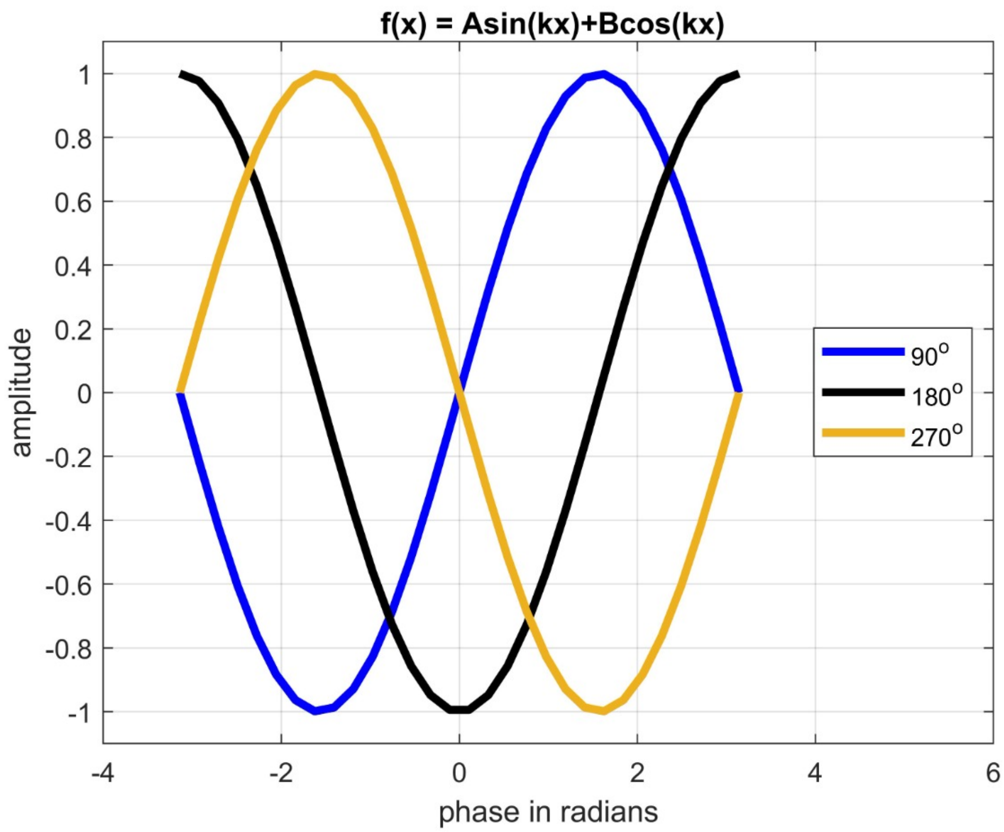
$$\begin{aligned} \frac{D}{Dt}(\overline{u} + u') &= \frac{\partial}{\partial t}(\overline{u} + u') + \overline{u}\frac{\partial u'}{\partial x} \\ &= \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x}, \end{aligned}$$

which is the original non-linear differential equation reduced to a linear differential equation. For such equations with constant coefficients (i.e., $\overline{u}$), the solutions are of a sinusoidal nature. In fact, when considering a section of an atmospheric wave along a latitude circle, this flow is equivalent to a sinusoidal shape.

A Fourier series represents a periodic function as a sum of sine and cosine waves. Consider the Fourier series

$$f(x) = A\sin(kx) + B\cos(kx),$$

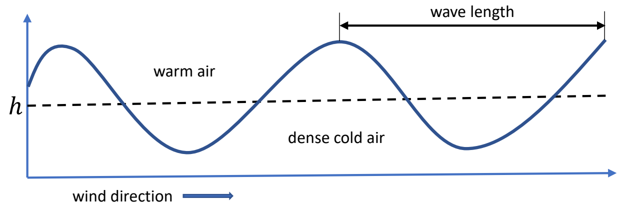
where A and B are real coefficients. The figure below shows the $f(x)$ results for the phase angles as shown.



The figure shows a propagating wave as oscillations for various phase angles that propagates in the positive direction. Atmospheric wave motions also are oscillations of variables such as pressure that propagate in space and time.

Shallow water gravity waves

In fluid dynamics, gravity waves are generated in a fluid or at the interface between two media, for example the atmosphere and the ocean or between cold and warm air layers. When a fluid element is displaced on such an interface, gravity attempts to restore the parcel towards equilibrium resulting in oscillations about the equilibrium state.

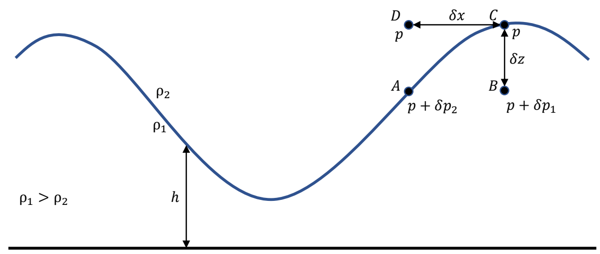


The restoring force is in the vertical, even in the case of horizontally propagating oscillations, such as in the presence of wind.

To mimic such as a mechanism consider a fluid in a channel. The back-and-forth oscillation of a lever or paddle. The net result is a sinusoidal disturbance which moves towards the right along the wind direction as shown in the figure above. Consider the example where the wavelengths of the waves are much greater then the depth of the fluid. This configuration leads to the vertical velocities to be small so that the hydrostatic approximation is valid:

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho g \\ \therefore \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) &= -\frac{\partial}{\partial x}\left(\rho g\right) \\ \therefore \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) &= -g\frac{\partial \rho}{\partial x} \end{aligned}$$

Consider a two-layer fluid system as shown in the figure below. The density in each of the two layers is constant, therefore $\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = 0$



Assume that there is no horizontal pressure gradient in the upper layer, so that the pressure, p, at points C and D are equal. For the pressure at points A and B we have, respectively,

$$p + \delta p_2 = p + \rho_2 g \delta z$$

and

$$p + \delta p_1 = p + \rho_1 g \delta z$$

where the hydrostatic equation was used once more and where the positive sign of both terms is a consequence of the pressure change and the height change acting in opposite directions.

Consider h, the depth of the lower layer. The change of h is equivalent to the change in height, therefore $\delta z \sim \delta h$.

$$\therefore \delta z = \frac{\partial h}{\partial x}\delta x.$$

Therefore

$$p + \delta p_2 = p + \rho_2 g \frac{\partial h}{\partial x}\delta x$$

and

$$p + \delta p_1 = p + \rho_1 g \frac{\partial h}{\partial x}\delta x.$$

The pressure gradient in the lower layer, and taking the limit $\delta x \to 0$:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \lim_{\delta x \to 0}\left[\frac{(p+\delta p_1) - (p + \delta p_2)}{\delta x}\right]\\ &= \lim_{\delta x \to 0}\left[\frac{g\frac{\partial h}{\partial x}(\rho_1 - \rho_2)\delta x}{\delta x}\right]\\ &= g\delta\rho\frac{\partial h}{\partial x}, \text{ with } \delta\rho = \rho_1 - \rho_2 \ (\rho_1 > \rho_2)\end{aligned}$$

With two dimensional motion in the x-, z-plane, the momentum equation for the lower layer is reduced to:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \text{ since } v = 0.$$

$$\therefore \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\left(g\delta\rho\frac{\partial h}{\partial x}\right)$$

The momentum equation for the lower layer becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

The continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ is reduced to

$$\frac{\partial u}{\partial x}=-\frac{\partial w}{\partial z}\ \text{ (since } v=0\text{)}$$

By integrating this equation vertically from the lower boundary ($z=0$) to the interface between the two layers ($z=h$), leads to

$$\begin{aligned}\int_0^h \frac{\partial w}{\partial z}dz &= -\int_0^h \frac{\partial u}{\partial x}dz\\ &= -\frac{\partial u}{\partial x}\int_0^h dz \text{ (if it is assumed that } u\neq u(z))\\ w(h)-w(0) &= -h\frac{\partial u}{\partial x}\end{aligned}$$

For a flat lower boundary $w(0)=0$; $w(h)$ is the rate at which the interface height is changing: $w(h)=\frac{Dh}{Dt}$

$$\begin{aligned}\frac{Dh}{Dt} &= \frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+w\frac{\partial h}{\partial z}\ \text{ (since } v=0\text{)}\\ &= \frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\ \text{ since the change of height at any point is constant}\end{aligned}$$

$$\begin{aligned}\therefore \frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} &= -h\frac{\partial u}{\partial x}\\ \therefore \frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu) &= 0\ \text{ (according to the chain rule of differentiation)}\end{aligned}$$

We have subsequently obtained a closed set of equations in the variables u and h:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{g\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

and

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0$$

Next, we expand these two variables in terms of their basic state and perturbation portions:

$$u=\overline{u}+u',\ \ h=H+h',$$

where $\overline{u}$ is a constant basic state zonal velocity, and H is the mean depth of the lower layer.

$$\therefore \frac{\partial}{\partial t}(\overline{u}+u')+(\overline{u}+u')\frac{\partial}{\partial x}(\overline{u}+u')+0+\frac{g\delta\rho}{\rho_1}\frac{\partial}{\partial x}(H+h')$$

assuming that $u\neq u(z)$. Using the linear perturbation method

$$\frac{\partial}{\partial t}(\overline{u}+u')=\frac{\partial u'}{\partial t},$$

$$(\overline{u}+u')\frac{\partial}{\partial x}(\overline{u}+u') = (\overline{u}+u')\frac{\partial u'}{\partial x} = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$
$$= \overline{u}\frac{\partial u'}{\partial x} \text{ since products of perturbations may be neglected}$$

and

$$\frac{g\delta\rho}{\rho_1}\frac{\partial}{\partial x}(H+h') = \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}.$$

$$\therefore \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + \frac{g\delta\rho}{\rho_1}\frac{\partial h'}{\partial x} = 0 \tag{1}$$

For $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0$:

$$\frac{\partial}{\partial t}(H+h') = \frac{\partial h'}{\partial t}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}(hu) &= \frac{\partial}{\partial x}\left[(H+h')(\overline{u}+u')\right]\\ &= \frac{\partial}{\partial x}\left[H\overline{u} + Hu' + \overline{u}h' + u'h'\right]\\ &= \frac{\partial}{\partial x}(Hu') + \frac{\partial}{\partial x}(\overline{u}h') \text{ since } H\overline{u} \text{ is a constant value and } u'h' \text{ is a product of perturbations.}\\ &= H\frac{\partial u'}{\partial x} + \overline{u}\frac{\partial h'}{\partial x}\end{aligned}$$

$$\therefore \frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} + \overline{u}\frac{\partial h'}{\partial x} = 0$$
$$\therefore \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)h' + H\frac{\partial u'}{\partial x} = 0 \tag{2}$$

Apply the operator $H\dfrac{\partial}{\partial x}$ to equation (1):

$$H\frac{\partial}{\partial x}\frac{\partial u'}{\partial t} + H\overline{u}\frac{\partial^2 u'}{\partial x^2} + \frac{Hg\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0 \tag{3}$$

Then apply the operator $\left(\dfrac{\partial}{\partial t} + \overline{u}\dfrac{\partial}{\partial x}\right)$ equation (2):

$$\therefore \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\frac{\partial h'}{\partial t} + \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\overline{u}\frac{\partial h'}{\partial x}\right) + \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(H\frac{\partial u'}{\partial x}\right) = 0$$

$$\therefore \frac{\partial^2 h'}{\partial t^2} + \overline{u}\frac{\partial}{\partial t}\frac{\partial h'}{\partial x} + \overline{u}\frac{\partial}{\partial t}\frac{\partial h'}{\partial x} + \overline{u}^2\frac{\partial^2 h'}{\partial x^2} + H\frac{\partial}{\partial x}\frac{\partial u'}{\partial t} + H\overline{u}\frac{\partial^2 u'}{\partial x^2} = 0$$
$$\therefore \frac{\partial^2 h'}{\partial t^2} + 2\overline{u}\frac{\partial}{\partial t}\frac{\partial h'}{\partial x} + \overline{u}^2\frac{\partial^2 h'}{\partial x^2} + H\frac{\partial}{\partial x}\frac{\partial u'}{\partial t} + H\overline{u}\frac{\partial^2 u'}{\partial x^2} = 0 \tag{4}$$

Subtract equation (3) from equation (4):

$$\frac{\partial^2 h'}{\partial t^2} + 2\overline{u}\frac{\partial}{\partial t}\frac{\partial h'}{\partial x} + \overline{u}^2\frac{\partial^2 h'}{\partial x^2} - \frac{Hg\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

$$\therefore \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 h' - \frac{Hg\delta\rho}{\rho_1}\frac{\partial^2 h'}{\partial x^2} = 0$$

which is a form of the standard wave equation with a solution representing a sinusoidal wave propagation in x. Such a solution can also be represented by an exponential function:

$$h' = Ae^{ik(x-ct)}$$

and only the real part has physical significance. A is a Fourier coefficient and c is the phase speed that we want to obtain.

Apply the following operators to the exponential function: $\frac{\partial^2}{\partial t^2}$, $\frac{\partial^2}{\partial t \partial x}$ and $\frac{\partial^2}{\partial x^2}$ and keeping in mind that $D_x\left(e^{g(x)}\right) = e^{g(x)}g'(x)$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}h' &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(Ae^{ik(x-ct)}\right)\right)\\ &= \frac{\partial}{\partial t}\left(Ae^{ik(x-ct)}(-ikc)\right)\\ &= (-ikc)^2 Ae^{ik(x-ct)}\\ &= -Ak^2c^2e^{ik(x-ct)} \text{ since } i^2 = -1\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t \partial x}h' &= \frac{\partial}{\partial t}\left(Ae^{ik(x-ct)}(ik)\right)\\ &= Aike^{ik(x-ct)}(-ikc)\\ &= Ak^2ce^{ik(x-ct)}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}h' &= \frac{\partial}{\partial x}\left(Ae^{ik(x-ct)}(ik)\right)\\ &= A(ik)^2e^{ik(x-ct)}\\ &= -Ak^2e^{ik(x-ct)}\end{aligned}$$

Therefore,

$$-Ak^2c^2e^{ik(x-ct)} + 2\overline{u}Ak^2ce^{ik(x-ct)} + \overline{u}^2\left(-Ak^2e^{ik(x-ct)}\right) - \frac{Hg\delta\rho}{\rho_1}\left(-Ak^2e^{ik(x-ct)}\right) = 0$$

$$\therefore -c^2 + 2\overline{u}c - \overline{u}^2 + \frac{Hg\delta\rho}{\rho_1} = 0$$

Solving for a quadratic equation in c:

$$
\begin{aligned}
c &= \frac{-(-2\overline{u}) \pm \left((-2\overline{u})^2 - 4\left(\overline{u}^2 - \dfrac{Hg\delta\rho}{\rho_1}\right)\right)^{1/2}}{2} \\
&= \overline{u} \pm \left(\frac{Hg\delta\rho}{\rho_1}\right)^{1/2}
\end{aligned}
$$

In the case of the upper layer air and the lower layer is water:

$$
\begin{aligned}
\delta\rho = \rho_1 - \rho_2 &\sim (1000 - 1)\,\mathrm{kg\,m^3} \text{ at 15°C} \\
&\sim 1000\,\mathrm{kg\,m^3}
\end{aligned}
$$

$$
\therefore \frac{\delta\rho}{\rho_1} \sim \frac{1000}{1000} = 1
$$

Therefore, the phase speed c simplifies to $\overline{u} \pm (Hg)^{1/2}$. $(Hg)^{1/2}$ is called the shallow water wave speed.

In the case of two atmospheric layers, $\dfrac{\delta\rho}{\rho_1} \sim 10^{-2}$. Therefore, the phase speed $c \simeq \overline{u} \pm (10^{-1}H)^{1/2}$ if the gravitational acceleration is taken as $10\,\mathrm{m\,s^{-2}}$. The shallow water wave speed then becomes $\left(\dfrac{H}{10}\right)^{1/2}$.

Shallow water wave speed for an ocean depth of $3\,\mathrm{km}$ the speed is $\sim 170\,\mathrm{m\,s^{-1}}$, while the shallow water wave speed in the atmosphere at the height of the $700\,\mathrm{hPa}$ level, i.e. $3\,\mathrm{km}$ above sea level, the speed is $\sim 17\,\mathrm{m\,s^{-1}}$. Long waves on the ocean surface travel very rapidly, while wave speeds for atmospheric waves travel at about one tenth the speed of ocean waves.

Rossby waves

The barotropic Rossby wave is an absolute vorticity conserving motion that exists due to the variations of the Coriolis parameter. Waves need to have a restoring force associated with them, and for Rossby waves that is the β-effect. The Coriolis force varies as you go north and south away from a reference latitude. This movement has a response on relative vorticity, causing deviating fluid parcels to be redirected back towards the reference latitude.

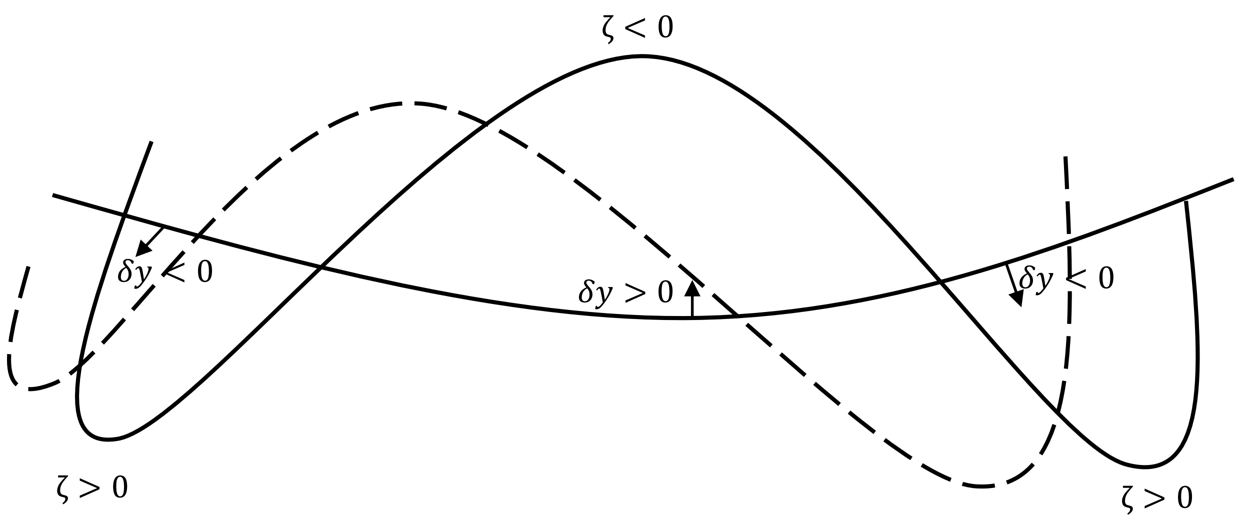


Figure 18: Solid wavy line shows the original perturbation position of a chain of fluid parcels. Dashed line shows westward displacement of the pattern to be explained later in this section.

Consider a closed chain of fluid parcels initially aligned along a constant circle of latitude. The absolute vorticity, which is the sum total of relative vorticity and the Coriolis parameter, or planetary vorticity, is conserved in large-scale Rossby waves. At time t_0, absolute vorticity is zero, and at a later time t_1, δy is the meridional displacement of a fluid parcel away from the initial circle of latitude. As a result, at time t_1

$$(\zeta + f)_{t_1} = (0 + f)_{t_0}$$
$$\therefore \zeta_{t_1} = f_{t_0} - f_{t_1} = -(f_{t_1} - f_{t_0}) = -\delta f$$

Since $\beta = \dfrac{df}{dy}$, $\delta f = \beta \delta y$, therefore $\zeta_{t_1} = -\beta \delta y$.

β is always positive and δy is negative (positive) for a parcel displacement to the north (south) of the reference latitude. Therefore, the perturbation vorticity will be positive (negative) for a southward (northward) displacement.

The horizontal distribution of geopotential at the $500\,\mathrm{hPa}$ level and the associated relative vorticity field are shown in Figure 19 for wavelengths and for two mean zonal flows. The figure is for Southern Hemisphere cases. Take note how the positive vorticity field corresponds with the $500\,\mathrm{hPa}$ ridges, and the negative vorticity field with the $500\,\mathrm{hPa}$ troughs.

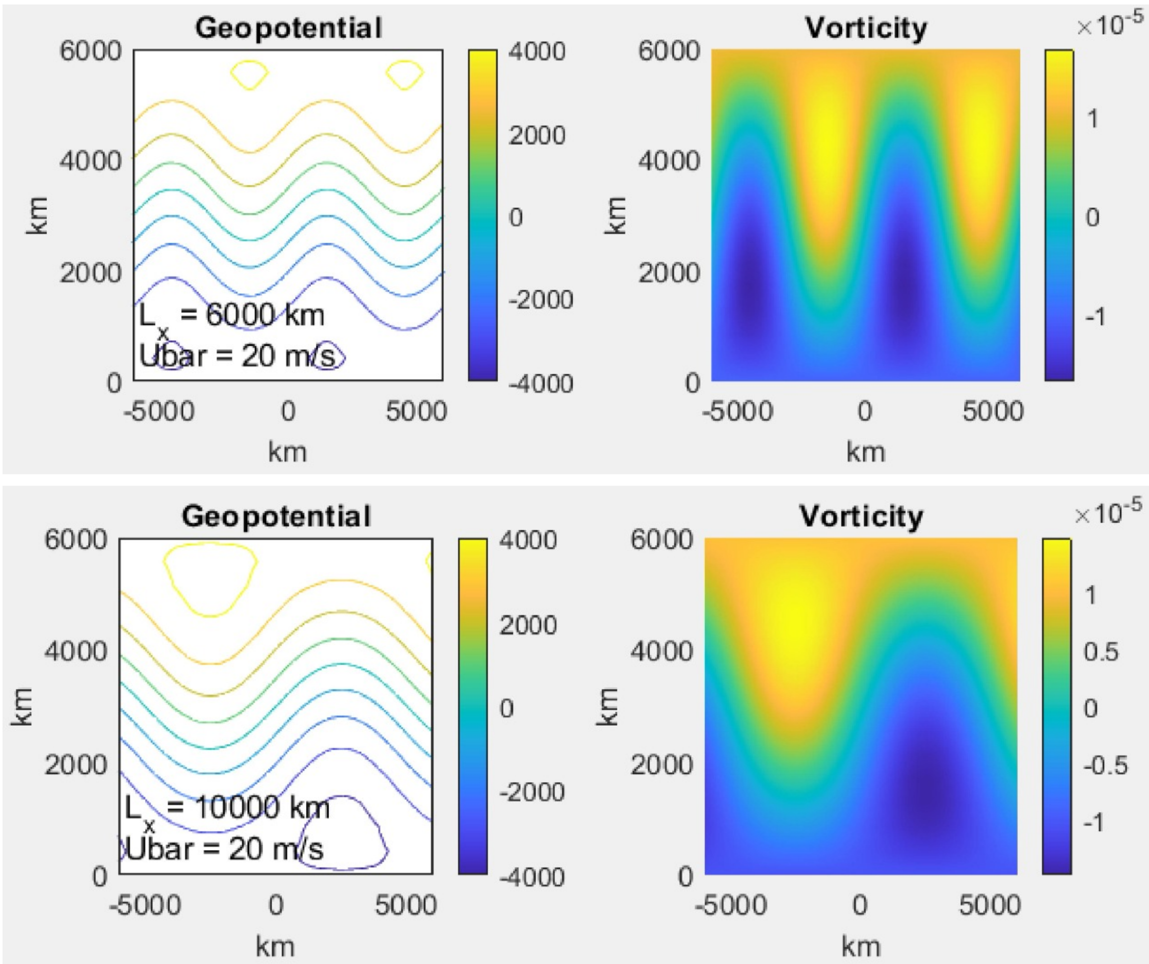


Figure 19: Geopotential heights at $500\,\mathrm{hPa}$ in units of m, and relative vorticity in units of $10^{-5}\,\mathrm{s}^{-1}$. The top panels show the results for a $6000\,\mathrm{km}$ wavelength, and the bottom panels show results for a $10000\,\mathrm{km}$ wavelength. The mean zonal flow for both cases is $20\,\mathrm{m\,s}^{-1}$.

The fluid parcels oscillate back and forth about their initial circle of latitude. Next, we want to complete the speed of wave propagation by considering δy to be represented as a sinusoidal meridional displacement:

$$\delta y = a \sin[k(x - ct)],$$

where a is the maximum displacement of the fluid parcels, c the wave speed, and k the zonal wave number. There is no zonal component so $u = 0$ and $\dfrac{\partial u}{\partial y} = 0$.

Meridional speed:

$$v = \frac{D}{Dy}\delta y = \frac{D}{Dy}\left(a\sin[k(x-ct)]\right)$$
$$= -kca\cos[k(x-ct)]$$

Relative vorticity:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-kca\cos[k(x-ct)]\right)$$
$$= k^2 ca\sin[k(x-ct)].$$

At t_1:

$$k^2 ca\sin[k(x-ct)] = -\beta a\sin[k(x-ct)]$$
$$\therefore c = -\frac{\beta}{k^2} < 0.$$

Therefore, the phase speed is westward relative to the mean flow.

Next we want to further investigate barotropic Rossby waves by finding solutions of the barotropic vorticity equation by applying the perturbation method. The Rossby wave is an absolute vorticity conserving motion. In purely horizontal flow in a fluid of constant depth (the divergence term vanishes) the barotropic vorticity equation is

$$\frac{D}{Dt}\left(\zeta_g + f\right) = 0.$$

Since absolute vorticity is conserved owing to the vanishing of the divergence of the horizontal wind, the flow is not required to be geostrophic. Therefore,

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = 0$$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + v\frac{\partial f}{\partial y} = 0$$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0$$

We next apply the perturbation method and assume that the motion is a result of a constant basic state zonal velocity plus a small horizontal perturbation, $u = \overline{u} + u'$, and a meridional perturbation as a result of the fluid parcels oscillating back and forth about a latitude circle. Therefore, $v = v'$ and $\zeta = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \zeta'$. The perturbation stream function is defined according to

$$u' = -\frac{\partial \psi'}{\partial y} \text{ and } v' = \frac{\partial \psi'}{\partial x}.$$

Then $\zeta' = \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi'}{\partial y}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi' = \nabla^2\psi'$.

Expanding on the barotropic vorticity equation

$$\left(\frac{\partial}{\partial t} + (\overline{u} + u')\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\nabla^2\psi' + \beta v' = 0.$$

Ignoring products of perturbation terms leads to

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0.$$

As before, we seek a solution of the form

$$\psi' = \text{Re}\left[\Psi e^{i(kx+ly-\nu t)}\right]$$

where k and l are wave numbers respectively in the zonal and meridional directions, ν is the angular frequency, and Ψ an amplitude coefficient.

Consider the terms of the perturbation barotropic vorticity equation separately:

First

$$\begin{aligned}\nabla^2\psi' &= \frac{\partial^2}{\partial x^2}\left(\Psi e^{i(kx+ly-\nu t)}\right) + \frac{\partial^2}{\partial y^2}\left(\Psi e^{i(kx+ly-\nu t)}\right)\\ &= (ik)^2\Psi e^{i(kx+ly-\nu t)} + (il)^2\Psi e^{i(kx+ly-\nu t)}\\ &= -(k^2+l^2)\Psi e^{i(kx+ly-\nu t)}\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\nabla^2\psi' = i\nu(k^2+l^2)\Psi e^{i(kx+ly-\nu t)}$$

and

$$\overline{u}\frac{\partial}{\partial x}\nabla^2\psi' = -ik\overline{u}(k^2+l^2)\Psi e^{i(kx+ly-\nu t)}$$

and

$$\beta\frac{\partial\psi'}{\partial x} = ik\beta\Psi e^{i(kx+ly-\nu t)}$$

The perturbation barotropic vorticity equation then reduces to:

$$\nu(k^2+l^2) - \overline{u}k(k^2+l^2) + k\beta = 0$$

$$\therefore \nu = \overline{u}k - \frac{\beta k}{(k^2+l^2)}$$

Since $c = \dfrac{\nu}{k}$,

$$c = \overline{u} - \frac{\beta}{(k^2+l^2)}$$

For the case of the mean wind, $\overline{u}$, vanishing and the meridional wave number, l, tending to zero,

$$c = -\frac{\beta}{k^2} < 0,$$

as before, always westward relative to the mean zonal flow.

The Rossby wave phase speed varies with k, and the wave is therefore dispersive. A zonal wave number can typically be defined as $k = 2\pi s/L$, with s (the planetary wave number) an integer number of waves around a latitude circle. For a typical mid-latitude synoptic-scale disturbance with $l \approx k$ (similar meridional and zonal scales) and a zonal wavelength of about $6000\,\mathrm{km}$

$$
\begin{aligned}
c - \overline{u} &= -\frac{\beta}{2k^2} = -\frac{\beta}{2\left(\dfrac{2\pi s}{L}\right)^2} = -\frac{\beta L^2}{8\pi^2 s^2} \\
&\sim \frac{-10^{-11}\,\mathrm{m^{-1}\,s^{-1}}(6\times 10^6\,\mathrm{m})^2}{8\pi^2 s^2} \\
&= -\frac{4.56}{s^2}\,\mathrm{m\,s^{-1}},
\end{aligned}
$$

which is the Rossby wave speed relative to the zonal flow. The Rossby wave phase speed therefore depends inversely on the square of the number of waves around a latitude circle, the phase speeds decrease rapidly with decreasing wavelength.

Take note that since the mean zonal wind is generally westerly, synoptic-scale Rossby waves usually move eastward if the zonal wind is stronger than the Rossby wave speed. However, the Rossby waves move eastward at a phase speed relative to the ground that is somewhat less than the mean zonal wind speed.

For longer wavelengths, say $L \sim 8\times 10^6\,\mathrm{m}$, the westward Rossby wave speed may be large enough so that the resulting disturbance is stationary. This implies that $\overline{u} = \dfrac{\beta}{2k^2}$, resulting in $\overline{u} = \dfrac{8.1}{s^2}\,\mathrm{m\,s^{-1}}$.

Since Rossby waves are dispersive (these waves may have different wavelengths that propagate at different phase speeds) the resulting downstream development of new disturbances is of interest to weather forecasting and consequently need to be discussed further. The group velocity, the velocity at which the observable disturbance propagates, is $c_{gx} = \dfrac{\partial \nu}{\partial k}$ for the zonal case. Therefore,

$$
\begin{aligned}
c_{gx} &= \frac{\partial}{\partial k}\left(\overline{u}k - \frac{\beta k}{(k^2+l^2)}\right) \\
&= \overline{u} - \beta\frac{\partial}{\partial k}\left[\frac{k}{(k^2+l^2)}\right]
\end{aligned}
$$

Consider the quotient rule of differentiation that states

$$
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\dfrac{d}{dx}\left(f(x)\right)g(x) - \dfrac{d}{dx}\left(g(x)\right)f(x)}{\left(g(x)\right)^2}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial k}\left[\frac{k}{(k^2+l^2)}\right] &= \frac{1\times(k^2+l^2) - 2k\times k}{(k^2+l^2)^2} \\
&= \frac{l^2-k^2}{(k^2+l^2)^2}
\end{aligned}
$$

$$\therefore c_{gx} = \overline{u} - \beta \frac{l^2 - k^2}{(k^2 + l^2)^2}$$

For stationary waves ($\overline{u} = 0$) and with diminishing meridional wave number l,

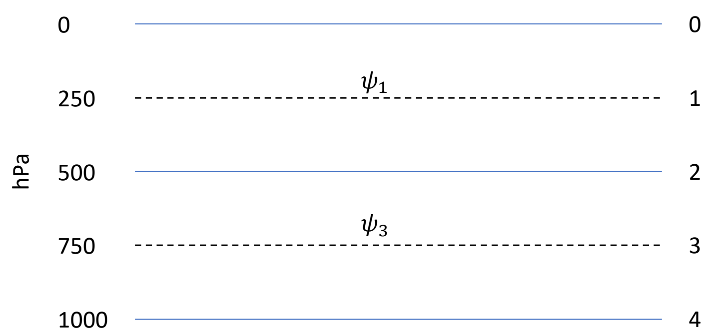
$$c_{gx} = \frac{\beta}{k^2} > 0,$$

therefore always has an eastward zonal component relative to the Earth. Since the wave energy propagates at the group velocity, Rossby wave energy propagation is downstream.

A two-layer model to understand baroclinic instability

Quasi-geostrophic theory accounts for the observed relationships between vorticity, temperature and vertical velocity in mid-latitude synoptic-scale systems. Lacking, however, is quantitative information on the origin, growth rates and propagating speeds. In this section we want to analyse the role of relative vorticity in the development of these systems. For this purpose we will introduce the concept of dynamic instability. This type of instability deals with waves in a moving fluid system such as the atmosphere.

Barotropic instability arises from vorticity distributions in a non-divergent, two dimensional flow – this instability is associated with horizontal shear. Such instabilities grow by converting kinetic energy from the mean flow between the air current and a perturbation. A necessary condition for barotropic instability is that the vorticity field has both positive and negative signs in the domain, as we have seen in the discussion on Rossby waves.



Baroclinic instability is associated with vertical shear arising from the existence of a meridional temperature gradient of a synoptic-scale disturbance. Such a system converts potential energy of the basic flow into kinetic energy of an induced perturbation.

In our analyses to follow on baroclinic instability we again assume that a small perturbation consisting of a small Fourier wave mode of the form $e^{ik(x-ct)}$ is introduced into the flow as explained through linear perturbation theory. First, we discuss a simple model that can incorporate baroclinic processes as shown in the figure above.

The atmosphere is represented by surfaces at the top of the troposphere ($0\,\mathrm{hPa}$; although at this pressure level we are actually at the stratosphere), middle troposphere ($500\,\mathrm{hPa}$) and at the surface ($1000\,\mathrm{hPa}$). Take note that although the South African land surface is mostly at approximately $850\,\mathrm{hPa}$, the mid-latitude systems most often affecting our weather are located on the surface at sea-level.

The quasi-geostrophic vorticity equation is applied at the $750\,\mathrm{hPa}$ and $250\,\mathrm{hPa}$ levels. The thermodynamic energy equation is applied at the $500\,\mathrm{hPa}$ level. We also define a geostrophic streamfunction, $\psi = \dfrac{\Phi}{f_0}$, and apply it at the same level where the vorticity equation is applied.

Recall the definition of the geostrophic wind $\vec{V}_g \equiv \dfrac{1}{f_0}\vec{k} \times \vec{\nabla}\Phi$, and geostrophic vorticity $\zeta_g = \dfrac{1}{f_0}\nabla^2\Phi$, and rewrite them in terms of the geostrophic streamfunction:

$$\vec{V}_\psi = \frac{1}{f_0}\vec{k} \times \vec{\nabla}(f_0\psi) \implies \vec{V}_\psi = \vec{k} \times \vec{\nabla}\psi$$

$$\zeta_g = \frac{1}{f_0}\nabla^2(f_0\psi) \implies \zeta_g = \nabla^2\psi$$

Consider the geostrophic vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f) + f_0\frac{\partial \omega}{\partial p}$$

$$\therefore \frac{\partial}{\partial t}(\nabla^2\psi) = -\vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi + f) + f_0\frac{\partial \omega}{\partial p}$$

Take note that $\vec{V}_\psi$ is still the geostrophic wind, but in terms of a streamfunction.

Consider

$$\begin{aligned}
\vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi + f) &= \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + (u_g\vec{i} + v_g\vec{j})_\psi \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right) f \\
&= \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + \beta v_g \\
&= \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + \beta\frac{1}{f_0}\frac{\partial \Phi}{\partial x} \\
&= \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + \beta\frac{\partial \psi}{\partial x}
\end{aligned}$$

$$\implies \frac{\partial}{\partial t}\nabla^2\psi + \vec{V}_\psi \cdot \vec{\nabla}(\nabla^2\psi) + \beta\frac{\partial \psi}{\partial x} = f_0\frac{\partial \omega}{\partial p}$$

At levels 1 ($250\,\mathrm{hPa}$) and 3 ($750\,\mathrm{hPa}$), the equation above at each level respectively is expressed as

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \vec{V}_1 \cdot \vec{\nabla}(\nabla^2\psi_1) + \beta\frac{\partial \psi_1}{\partial x} = f_0\left(\frac{\partial \omega}{\partial p}\right)_1$$

and

$$\frac{\partial}{\partial t}\nabla^2\psi_3 + \vec{V}_3 \cdot \vec{\nabla}(\nabla^2\psi_3) + \beta\frac{\partial \psi_3}{\partial x} = f_0\left(\frac{\partial \omega}{\partial p}\right)_3.$$

The divergence term on the right-hand side of these equations can be estimated using finite difference approximations:

$$\left(\frac{\partial \omega}{\partial p}\right)_1 \approx \frac{\omega_2 - \omega_0}{\delta p} \quad \text{and} \quad \left(\frac{\partial \omega}{\partial p}\right)_3 \approx \frac{\omega_4 - \omega_2}{\delta p}.$$

In each case, $\delta p = 500\,\mathrm{hPa}$. We assume that there is no upward motion on the top of the two-layer model, so that $\omega_0 = 0$, and also assume that $\omega_4 \approx 0$ and the surface boundary of the lower layer. So, at levels 1 and 3 the two equations become

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \vec{V_1}\cdot\vec{\nabla}(\nabla^2\psi_1) + \beta\frac{\partial\psi_1}{\partial x} = \frac{f_0}{\delta p}\omega_2$$

and

$$\frac{\partial}{\partial t}\nabla^2\psi_3 + \vec{V_3}\cdot\vec{\nabla}(\nabla^2\psi_3) + \beta\frac{\partial\psi_3}{\partial x} = -\frac{f_0}{\delta p}\omega_2.$$

Next, we consider the thermodynamic energy equation at $500\,\mathrm{hPa}$ where the geostrophic wind is again represented by $\vec{V_\psi}$:

$$\left(\frac{\partial}{\partial t} + \vec{V_\psi}\cdot\vec{\nabla}\right)\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{RJ}{c_p p} = 0$$

if it is assumed that the diabatic heating term, J, can be neglected.

$$\therefore \frac{\partial}{\partial t}\left(-\frac{\partial}{\partial p}(f_0\psi)\right) = -\vec{V_\psi}\cdot\vec{\nabla}\left(-\frac{\partial}{\partial p}(f_0\psi)\right) + \sigma\omega$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial p}\right) = -\vec{V_\psi}\cdot\vec{\nabla}\left(\frac{\partial\psi}{\partial p}\right) - \frac{\sigma}{f_0}\omega.$$

We will next, as before, use a finite difference approximation to evaluate $\dfrac{\partial\psi}{\partial p}$:

$$\left(\frac{\partial\psi}{\partial p}\right) \approx \frac{\psi_3 - \psi_1}{\delta p}$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\psi_3 - \psi_1}{\delta p}\right) = -\vec{V_\psi}\cdot\vec{\nabla}\left(\frac{\psi_3 - \psi_1}{\delta p}\right) - \frac{\sigma}{f_0}\omega.$$

Since this form of the thermodynamic energy equation is applied at level 2 ($500\,\mathrm{hPa}$), the equation becomes:

$$\frac{\partial}{\partial t}\left(\frac{\psi_3 - \psi_1}{\delta p}\right) = -\vec{V_2}\cdot\vec{\nabla}\left(\frac{\psi_3 - \psi_1}{\delta p}\right) - \frac{\sigma}{f_0}\omega_2$$

$$\therefore \frac{\partial}{\partial t}\left(\psi_1 - \psi_3\right) = -\vec{V_2}\cdot\vec{\nabla}\left(\psi_1 - \psi_3\right) + \frac{\sigma\delta p}{f_0}\omega_2$$

With the first term on the right the advection of the 250 - $750\,\mathrm{hPa}$ thickness by the geostrophic wind at $500\,\mathrm{hPa}$.

To summarise, we have developed a set of prediction equations in the variables ψ_1, ψ_3 and ω_2:

$$\frac{\partial}{\partial t}\nabla^2\psi_1 = -\vec{V_1}\cdot\vec{\nabla}(\nabla^2\psi_1) - \beta\frac{\partial\psi_1}{\partial x} + \frac{f_0}{\delta p}\omega_2$$

$$\frac{\partial}{\partial t}\nabla^2\psi_3 = -\vec{V_3}\cdot\vec{\nabla}(\nabla^2\psi_3) - \beta\frac{\partial\psi_3}{\partial x} - \frac{f_0}{\delta p}\omega_2.$$

$$\frac{\partial}{\partial t}\left(\psi_1 - \psi_3\right) = -\vec{V_2}\cdot\vec{\nabla}\left(\psi_1 - \psi_3\right) + \frac{\sigma\delta p}{f_0}\omega_2$$

Next, we apply linear perturbation analysis to this set of equations, but first we need to define streamfunctions that consist of basic state parts and perturbations. We select

$$\psi_1 = -U_1 y + \psi_1'(x,t) \text{ and } \psi_3 = -U_3 y + \psi_3'(x,t).$$

Since

$$\vec{V_\psi} = \vec{k} \times \vec{\nabla}\psi = -\frac{\partial \psi}{\partial y}\vec{i} + \frac{\partial \psi}{\partial x}\vec{j}$$

$$\therefore u_\psi = -\frac{\partial \psi}{\partial y} \text{ and } v_\psi = \frac{\partial \psi}{\partial x}$$

$$\therefore u_{\psi_1} = U_1 \text{ and } u_{\psi_3} = U_3.$$

Therefore, the zonal velocities at levels 1 and 3 respectively are U_1 and U_3.

For v_ψ we have $v_{\psi_1} = \dfrac{\partial \psi_1'}{\partial x}$ and $v_{\psi_3} = \dfrac{\partial \psi_3'}{\partial x}$. Therefore the perturbation parts also represent meridional velocities.

We also include a vertical velocity component represented by $\omega_2 = \omega_2'(x,t)$.

We will next substitute the perturbation equations into the set of three prediction equations. First consider the vorticity equation at level 1:

$$\underbrace{\frac{\partial}{\partial t}\nabla^2\psi_1}_{\text{Term A}} = \underbrace{-\vec{V_1}\cdot\vec{\nabla}(\nabla^2\psi_1)}_{\text{Term B}} \underbrace{-\beta\frac{\partial \psi_1}{\partial x}}_{\text{Term C}} + \underbrace{\frac{f_0}{\delta p}\omega_2}_{\text{Term D}}$$

Term A:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2\psi_1 &= \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(-U_1 y + \psi_1'\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial^2 \psi_1'}{\partial x^2}\right)\end{aligned}$$

Term B:

$$\begin{aligned}-\vec{V_1}\cdot\vec{\nabla}(\nabla^2\psi_1) &= -\left(U_1\vec{i} + \frac{\partial \psi_1'}{\partial x}\vec{j}\right)\cdot\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_1'\\ &= -U_1\frac{\partial}{\partial x}\frac{\partial^2 \psi_1'}{\partial x^2}\end{aligned}$$

Term C:

$$-\beta\frac{\partial \psi_1}{\partial x} = -\beta\frac{\partial \psi_1'}{\partial x}$$

Term D:

$$\frac{f_0}{\delta p}\omega_2 = \frac{f_0}{\delta p}\omega_2'$$

$$\Longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial^2 \psi_1'}{\partial x^2}\right) = -U_1\frac{\partial}{\partial x}\frac{\partial^2 \psi_1'}{\partial x^2} - \beta\frac{\partial \psi_1'}{\partial x} + \frac{f_0}{\delta p}\omega_2'$$

$$\therefore \left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right) \frac{\partial^2 \psi_1'}{\partial x^2} + \beta \frac{\partial \psi_1'}{\partial x} = \frac{f_0}{\delta p} \omega_2'$$

And similarly for level 3:

$$\left(\frac{\partial}{\partial t} + U_3 \frac{\partial}{\partial x}\right) \frac{\partial^2 \psi_3'}{\partial x^2} + \beta \frac{\partial \psi_3'}{\partial x} = -\frac{f_0}{\delta p} \omega_2'$$

Revisiting the thermodynamic energy equation at level 2:

$$\underbrace{\frac{\partial}{\partial t} (\psi_1 - \psi_3)}_{\text{Term A}} = \underbrace{-\vec{V_2} \cdot \vec{\nabla} (\psi_1 - \psi_3)}_{\text{Term B}} + \underbrace{\frac{\sigma \delta p}{f_0} \omega_2}_{\text{Term C}}$$

Term A:

$$\frac{\partial}{\partial t} (\psi_1 - \psi_3) = \frac{\partial}{\partial t} (\psi_1' - \psi_3')$$

Term B:

$$-\vec{V_2} \cdot \vec{\nabla} (\psi_1 - \psi_3) = -\left(-\frac{\partial \psi_2}{\partial y} \vec{i} + \frac{\partial \psi_2}{\partial x} \vec{j}\right) \cdot \vec{\nabla} (\psi_1 - \psi_3),$$

since $\vec{V_\psi} = \vec{k} \times \vec{\nabla} \psi$. ψ_2 is the $500\,\mathrm{hPa}$ streamfunction and not a predicted field in this model. Therefore, ψ_1 has to be obtained by linearly interpolating between the $250\,\mathrm{hPa}$ and $750\,\mathrm{hPa}$ levels:

$$\psi_2 = \frac{1}{2} (\psi_1 + \psi_3).$$

Therefore

$$\begin{aligned} \frac{\partial \psi_2}{\partial y} &= \frac{1}{2} \frac{\partial}{\partial y} \left(-U_1 y + \psi_1' - U_3 y + \psi_3'\right) \\ &= -\frac{1}{2}(U_1 + U_3) \\ &= -U_m, \end{aligned}$$

the vertically averaged mean zonal wind, with $U_m \equiv \dfrac{1}{2}(U_1 + U_3)$.

Consider

$$\begin{aligned} \frac{\partial \psi_2}{\partial x} &= \frac{1}{2} \frac{\partial}{\partial x} \left(-U_1 y + \psi_1' - U_3 y + \psi_3'\right) \\ &= -\frac{1}{2} \left(\frac{\partial \psi_1'}{\partial x} + \frac{\partial \psi_3'}{\partial x}\right) \end{aligned}$$

Next consider

$$\begin{aligned} \vec{\nabla}(\psi_1 - \psi_3) &= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j}\right) \left(-U_1 y + \psi_1' + U_3 y - \psi_3'\right) \\ &= \frac{\partial}{\partial x} \left(\psi_1' - \psi_3'\right) \vec{i} + (-U_1 + U_3) \vec{j} \\ &= \frac{\partial}{\partial x} \left(\psi_1' - \psi_3'\right) \vec{i} - 2U_T \vec{j}, \end{aligned}$$

with $U_T \equiv \frac{1}{2}(U_1 - U_3)$, the vertically averaged mean thermal wind.

Term B then becomes:

$$-\left(U_m\vec{i} + \frac{1}{2}\left(\frac{\partial\psi_1'}{\partial x} + \frac{\partial\psi_3'}{\partial x}\right)\vec{j}\right)\cdot\left(\frac{\partial}{\partial x}\left(\psi_1' - \psi_3'\right)\vec{i} - 2U_T\vec{j}\right) = -U_m\left(\frac{\partial\psi_1'}{\partial x} - \frac{\partial\psi_3'}{\partial x}\right) + U_T\left(\frac{\partial\psi_1'}{\partial x} + \frac{\partial\psi_3'}{\partial x}\right)$$

Term C:

$$\frac{\sigma\delta p}{f_0}\omega_2 = \frac{\sigma\delta p}{f_0}\omega_2'$$

Then put all the terms together again to produce

$$\frac{\partial}{\partial t}\left(\psi_1' - \psi_3'\right) = -U_m\left(\frac{\partial\psi_1'}{\partial x} - \frac{\partial\psi_3'}{\partial x}\right) + U_T\left(\frac{\partial\psi_1'}{\partial x} + \frac{\partial\psi_3'}{\partial x}\right) + \frac{\sigma\delta p}{f_0}\omega_2'$$

$$\therefore \left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)\left(\psi_1' - \psi_3'\right) - U_T\frac{\partial}{\partial x}\left(\psi_1' + \psi_3'\right) = \frac{\sigma\delta p}{f_0}\omega_2'$$

We also want to express the vorticity equation in terms of U_m and U_T:

$$U_m + U_T = \frac{1}{2}(U_1 + U_3) + \frac{1}{2}(U_1 - U_3) = U_1$$
$$U_m - U_T = U_3$$

The vorticity equation at level 1 then becomes

$$\left[\frac{\partial}{\partial t} + (U_m + U_T)\frac{\partial}{\partial x}\right]\frac{\partial^2\psi_1'}{\partial x^2} + \beta\frac{\partial\psi_1'}{\partial x} = \frac{f_0}{\delta p}\omega_2',$$

and at level 3:

$$\left[\frac{\partial}{\partial t} + (U_m - U_T)\frac{\partial}{\partial x}\right]\frac{\partial^2\psi_3'}{\partial x^2} + \beta\frac{\partial\psi_3'}{\partial x} = -\frac{f_0}{\delta p}\omega_2'.$$

In order to eliminate ω_2' between these two vorticity equations, we define barotropic perturbations by $\psi_m \equiv \frac{1}{2}(\psi_1' + \psi_3')$ and baroclinic perturbations by $\psi_T \equiv \frac{1}{2}(\psi_1' - \psi_3')$.

Adding the two vorticity equations together leads to

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_1'}{\partial x^2} + (U_m + U_T)\frac{\partial}{\partial x}\frac{\partial^2\psi_1'}{\partial x^2} + \frac{\partial}{\partial t}\frac{\partial^2\psi_3'}{\partial x^2} + (U_m - U_T)\frac{\partial}{\partial x}\frac{\partial^2\psi_3'}{\partial x^2} + \beta\frac{\partial}{\partial x}(\psi_1' + \psi_3') = 0$$

$$\therefore \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}(\psi_1' + \psi_3') + U_m\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2} + \frac{\partial^2\psi_3'}{\partial x^2}\right) + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2} - \frac{\partial^2\psi_3'}{\partial x^2}\right) + \beta\frac{\partial}{\partial x}(\psi_1' + \psi_3') = 0$$

$$\therefore \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\psi_m + U_m\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}\psi_m + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_T}{\partial x^2}\right) + \beta\frac{\partial}{\partial x}\psi_m = 0$$

$$\therefore \left[\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial x^2}\psi_m + \beta\frac{\partial}{\partial x}\psi_m + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_T}{\partial x^2}\right) = 0$$

Subtracting the two vorticity equations leads to

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_1'}{\partial x^2}+(U_m+U_T)\frac{\partial}{\partial x}\frac{\partial^2\psi_1'}{\partial x^2}-\frac{\partial}{\partial t}\frac{\partial^2\psi_3'}{\partial x^2}-(U_m-U_T)\frac{\partial}{\partial x}\frac{\partial^2\psi_3'}{\partial x^2}+\beta\frac{\partial}{\partial x}(\psi_1'-\psi_3')=\frac{f_0}{\delta p}\omega_2'$$

$$\therefore \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\psi_T+U_T\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}\psi_m+\beta\frac{\partial}{\partial x}\psi_T=\frac{f_0}{\delta p}\omega_2'+U_m\frac{\partial}{\partial x}\frac{\partial^2\psi_T}{\partial x^2}$$

$$\therefore \left[\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial x^2}\psi_T+U_T\frac{\partial}{\partial x}\frac{\partial^2\psi_m}{\partial x^2}+\beta\frac{\partial}{\partial x}\psi_T=\frac{f_0}{\delta p}\omega_2'$$

Recall that we have derived an equation that can be used to eliminate ω_2':

$$\left(\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right)(\psi_1'-\psi_3')-U_T\frac{\partial}{\partial x}(\psi_1'+\psi_3')=\frac{\sigma\delta p}{f_0}\omega_2'$$

$$\therefore \omega_2'=\frac{f_0}{\sigma\delta p}\left[\left(\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right)(2\psi_T)-U_T\frac{\partial}{\partial x}(2\psi_m)\right]$$

Now we can eliminate ω_2':

$$\left[\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial x^2}\psi_T+U_T\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}\psi_m+\beta\frac{\partial}{\partial x}\psi_T=\frac{2f_0^2}{\sigma\delta p^2}\left[\left(\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right)\psi_T-U_T\frac{\partial}{\partial x}\psi_m\right]$$

$$\therefore \left[\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right]\left(\frac{\partial^2}{\partial x^2}\psi_T-\frac{2f_0^2}{\sigma\delta p^2}\psi_T\right)+\beta\frac{\partial}{\partial x}\psi_T+U_T\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\psi_m+\frac{2f_0^2}{\sigma\delta p^2}\psi_m\right)=0$$

We have managed to derive two equations that respectively govern the evolution of the barotropic (vertically averaged) perturbation vorticity

$$\left[\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial x^2}\psi_m+\beta\frac{\partial}{\partial x}\psi_m+U_T\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\psi_T\right)=0,$$

and the evolution of the baroclinic (thermal) perturbation vorticity

$$\left[\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right]\left(\frac{\partial^2}{\partial x^2}\psi_T-2\lambda^2\psi_T\right)+\beta\frac{\partial}{\partial x}\psi_T+U_T\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\psi_m+2\lambda^2\psi_m\right)=0,$$

where $\lambda^2\equiv\dfrac{f_0^2}{\sigma\delta p^2}$.

As we have done before, we assume that wave-like solutions exist in the form

$$\psi_m=Ae^{ik(x-ct)},\psi_T=Be^{ik(x-ct)}$$

First, we will apply the relevant differential operators to ψ_m and ψ_T.

$$\frac{\partial}{\partial t}\psi_m = (-ikc)Ae^{ik(x-ct)}$$
$$\frac{\partial}{\partial t}\psi_T = (-ikc)Be^{ik(x-ct)}$$
$$\frac{\partial}{\partial x}\psi_m = (ik)Ae^{ik(x-ct)}$$
$$\frac{\partial}{\partial x}\psi_T = (ik)Be^{ik(x-ct)}$$
$$\frac{\partial^2}{\partial x^2}\psi_m = (ik)^2Ae^{ik(x-ct)}$$
$$\frac{\partial^2}{\partial x^2}\psi_T = (ik)^2Be^{ik(x-ct)}$$
$$\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}\psi_m = (ik)^3Ae^{ik(x-ct)}$$
$$\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}\psi_T = (ik)^3Be^{ik(x-ct)}$$
$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\psi_m = (-ikc)(ik)^2Ae^{ik(x-ct)}$$
$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\psi_T = (-ikc)(ik)^2Be^{ik(x-ct)}$$

The barotropic perturbation equation is reduced to

$$(-ikc)(ik)^2A + U_m(ik)^3A + \beta(ik)A + U_T(ik)^3B = 0$$
$$\therefore ik\left[ck^2A - U_mk^2A + \beta A\right] - ik^3U_TB = 0$$
$$\therefore ik\left[(c-U_m)k^2 + \beta\right]A - ik^3U_TB = 0$$

The baroclinic perturbation equation is reduced to

$$(-ikc)(ik)^2B - 2\lambda^2(-ikc)B + U_m(ik)^3B - 2\lambda^2U_m(ik)B + \beta(ik)B + U_T(ik^3)A + 2\lambda^2U_T(ik)A = 0$$
$$\therefore ik\left[ck^2B + 2\lambda^2cB - k^2U_mB - 2\lambda^2U_mB + \beta B\right] + ik\left[-k^2U_TA + 2\lambda^2U_TA\right] = 0$$
$$\therefore ik\left[(c-U_m)(k^2+2\lambda^2) + \beta\right]B - ikU_T(k^2-2\lambda^2)A = 0$$

We obtain a pair of simulations linear algebraic equations for the coefficients A and B (after dividing throughout by ik):

$$\left[(c-U_m)k^2 + \beta\right]A - k^2U_TB = 0$$
$$-U_T(k^2-2\lambda^2)A + \left[(c-U_m)(k^2+2\lambda^2) + \beta\right]B = 0$$

For this set of equations, non-trivial solutions will exist only if the determinant of the coefficients A and B is zero.

$$\begin{vmatrix} (c-U_m)k^2 + \beta & -k^2U_T \\ -U_T(k^2-2\lambda^2) & (c-U_m)(k^2+2\lambda^2) + \beta \end{vmatrix} = 0$$

$$\therefore ((c - U_m)k^2 + \beta)((c - U_m)(k^2 + 2\lambda^2) + \beta) - k^2 U_T^2 (k^2 - 2\lambda^2) = 0$$
$$\therefore k^2(k^2 + 2\lambda^2)(c - U_m)^2 + 2\beta(k^2 + \lambda^2)(c - U_m) + \beta^2 + U_T^2 k^2 (k^2 - 2\lambda^2) = 0$$

Here is another relationship in which the phase speed, c, is dependent on the zonal wave number, k, thus making it a dispersion relationship. To solve this quadratic relationship we have

$$\begin{aligned} c - U_m &= \frac{-2\beta(k^2+\lambda^2) \pm \left[\left(2\beta(k^2+\lambda^2)\right)^2 - 4\left(k^2(k^2+2\lambda^2)\right)\left(\beta^2 + U_T^2 k^2(2\lambda^2 - k^2)\right)\right]^{1/2}}{2k^2(k^2+2\lambda^2)} \\ &= -\frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \left[\frac{\beta^2\lambda^4 - U_T^2 k^4 (k^2+2\lambda^2)(2\lambda^2 - k^2)}{k^4(k^2+2\lambda^2)^2}\right]^{1/2} \\ &= -\frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \delta^{1/2} \end{aligned}$$

with

$$\begin{aligned} \delta &= \frac{\beta^2\lambda^4 - U_T^2 k^4 (k^2+2\lambda^2)(2\lambda^2 - k^2)}{k^4(k^2+2\lambda^2)^2} \\ &= \frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} - \frac{U_T^2(2\lambda^2 - k^2)}{k^2+2\lambda^2} \end{aligned}$$

The proposed wave-like forms are solutions to the system that governs the evolution of the barotropic and baroclinic vortices, on condition that the phase speed, c has the above solution. At first glance of this solution, the phase speed has an imaginary component if $\delta < 0$, consequently leading to the perturbations amplifying exponentially. However, in the special case of $U_T = 0$ (the basic state zonal wind vanishes and $U_1 = U_3$ as a result) and the mean flow is barotropic (the mean thermal wind at the two levels is the same), $\delta > 0$ since $\dfrac{\beta^2\lambda^4}{k^2(k^4+2\lambda^2)^2}$ must be positive.

Two phase speeds, c_1 and c_2 respectively, are obtained as a result of $U_T = 0$.

$$c_1 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} + \delta^{1/2}$$

and

$$c_2 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} - \delta^{1/2}$$

$$\begin{aligned} c_1 &= U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} + \left(\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2}\right)^{1/2} \\ &= U_m - \frac{\beta k^2 - \beta\lambda^2 + \beta\lambda^2}{k^2(k^2+2\lambda^2)} \\ &= U_m - \beta(k^2+2\lambda^2)^{-1}, \end{aligned}$$

with

$$c_2 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} = U_m - \beta k^{-2}.$$

The solution for c_2 should look familiar since it is the dispersion relationship for a barotropic Rossby wave. Revisiting the pair of simultaneous equations for the coefficients of the wave-like solutions for, respectively, the barotropic and baroclinic perturbations and substituting the phase speed, c, by $U_m - \beta k^{-2}$ results in

$$-U_T(k^2 - 2\lambda^2)A + \left[(U_m - \beta k^{-2} - U_m)(k^2 + 2\lambda^2) + \beta\right] B = 0$$
$$\text{Since } U_T = 0, \; -2\beta\lambda^2 B = 0$$
$$\therefore B = 0$$

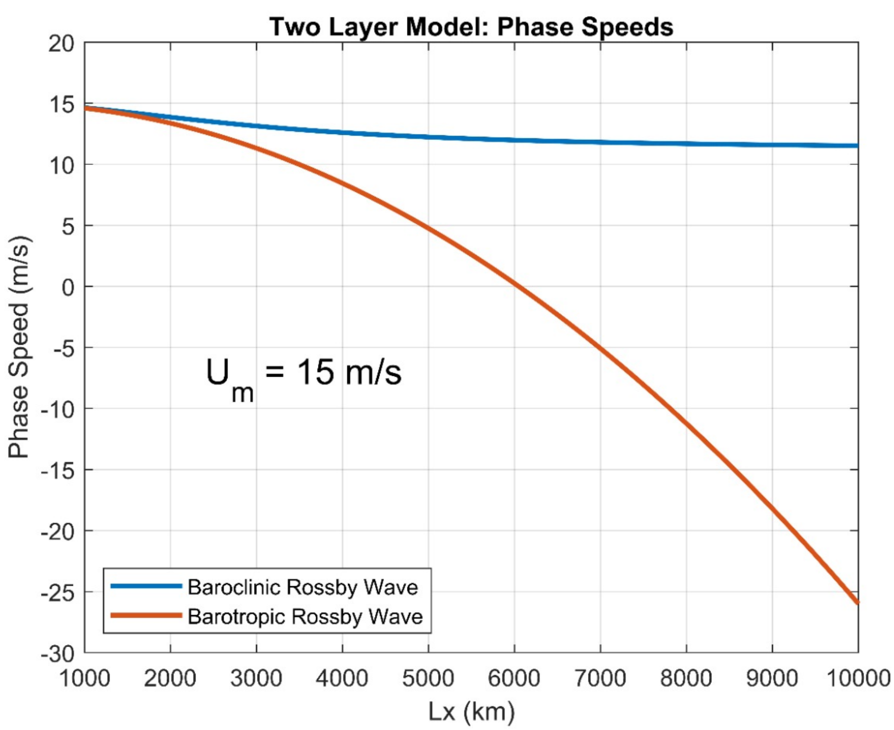
Since $B = 0$, $\psi_T = 0$ so that the perturbation is indeed barotropic in structure $(\psi_1' = \psi_3')$.

When using the phase speed c_2 (for an internal baroclinic Rossby wave), we get

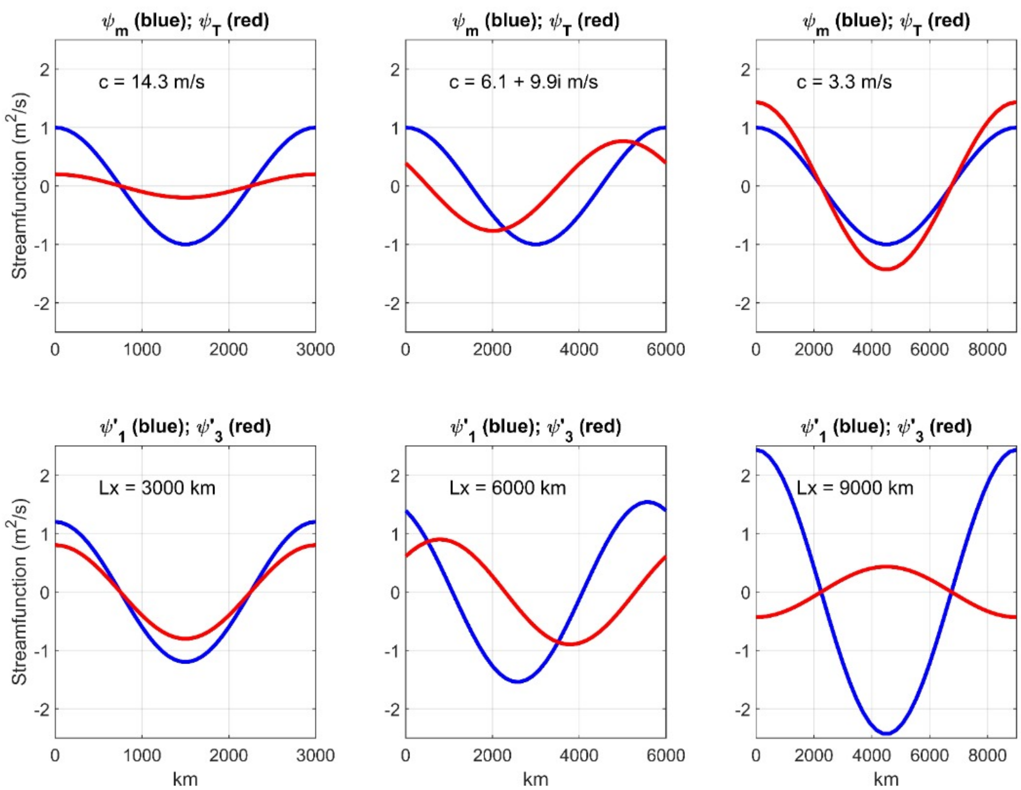
$$\left[(U_m - \beta(k^2 + 2\lambda^2)^{-1} - U_m)k^2 + \beta\right] A - k^2 U_T B = 0$$
$$\text{Since } U_T = 0, \, (\beta k^2(k^2 + 2\lambda^2)^{-1} + \beta)A = 0$$
$$\therefore A = 0$$

Since $A = 0$, $\psi_m = 0$ so that the perturbation is baroclinic in structure even though the mean flow is barotropic $(U_T = 0)$. In fact, for the case of phase speed c_1, $\psi_1' \simeq -\psi_3'$, which means that the perturbation fields at, respectively, the 250 and $750\,\mathrm{hPa}$ levels are $180°$ out of phase.

By using typical parameters for average mid-latitude tropospheric conditions ($\sigma \simeq 2\times10^{-6}\,\mathrm{N^{-2}\,m^6\,s^{-2}}$, $\lambda^2 \simeq 2\times10^{-12}\,\mathrm{m^{-2}}$) we calculate the phase speeds for, respectively, baroclinic Rossby waves (c_2) and barotropic Rossby waves (c_1). We assume that the mean zonal wind, U_m, is equal to $15\,\mathrm{m\,s^{-1}}$. From the figure below it is clear that the calculated phase speeds are always less than U_m, so that both the barotropic and baroclinic disturbances move westward relative to the mean wind. Furthermore, for a barotropic disturbance long wavelengths cause the phase speeds to be strong westward – a scenario that is not found in nature.



In order to gain further insight into the effect of varying zonal wavelengths (L_x) on baroclinic (ψ_T) and barotropic (ψ_m) perturbation streamfunctions, we again consider their respective wave-like solutions. Also, we want to investigate the streamfunction perturbations of, respectively, the $250\,\mathrm{hPa}$ level $(\psi_1' = \psi_m + \psi_T)$ and at the $750\,\mathrm{hPa}$ level $(\psi_3' = \psi_m - \psi_T)$. The figure below shows the result for the three wavelengths of, respectively, $3000\,\mathrm{km}$, $6000\,\mathrm{km}$ and $9000\,\mathrm{km}$. Only for the $6000\,\mathrm{km}$ wavelength the phase speed has an imaginary component $(c = 6.1 + 9.9i\,\mathrm{m\,s^{-1}})$, which leads to the perturbation amplifying exponentially. In fact, for the unstable mode associated with the $6000\,\mathrm{km}$ wavelength, the perturbation amplitude increases to four times the initial amplitude after about 40 hours. Take note that the barotropic and baroclinic streamfunctions are not synchronised as can be seen on the figure for this wavelength. However, at a wavelength of $9000\,\mathrm{km}$ these two streamfunctions are in phase, but the streamfunction perturbations are $180°$ out of phase (i.e., $\psi_1' = -\psi_3'$, which means $\psi_m = 0$) so that the perturbation has a baroclinic structure.



The forcing of vertical motion in a quasi-geostrophic system as described by the two-level model can be expressed in terms of the sum of the forcing by thermal advection (level 2) plus the differential vorticity advection. By the latter is meant that the evaluation is done by the difference between the vorticity advection at level 1 and that at level 3. Next we evaluate the forcing of vertical motion in terms of the divergence of the Q-vector. Recall that

$$\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -2\vec{\nabla}\cdot\vec{Q} + f_0\beta\frac{\partial v_g}{\partial p}$$

with $\vec{Q} = -\frac{R}{p}\frac{\partial\vec{V}_g}{\partial x}\cdot\vec{\nabla}T\vec{i} - \frac{R}{p}\frac{\partial\vec{V}_g}{\partial y}\cdot\vec{\nabla}T\vec{j}$.

First consider

$$
\begin{aligned}
\frac{\partial^2\omega}{\partial p^2} &= \frac{(\partial\omega/\partial p)_3 - (\partial\omega/\partial p)_1}{\delta p} \\
&= \frac{(\omega_4-\omega_2)/\delta p - (\omega_2-\omega_0)/\delta p}{\delta p} \\
&= -\frac{2\omega_2}{\delta p^2} \text{ since } \omega_0 = \omega_4 = 0.
\end{aligned}
$$

The left-hand side of the omega equation then becomes

$$
\begin{aligned}
\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} &= \sigma\nabla^2\omega + f_0^2\left(-\frac{2\omega_2}{\delta p^2}\right) \\
&= \sigma\nabla^2\omega + \lambda^2\sigma\delta p^2\left(-\frac{2\omega_2}{\delta p^2}\right) \\
&= \sigma\left(\nabla^2 - 2\lambda^2\right)\omega_2 \\
&= \sigma\left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2 \text{ since the perturbation function is independent on } y.
\end{aligned}
$$

To transform the right-hand side of the omega equation we first consider the hydrostatic equation

$$
\begin{aligned}
\frac{RT}{p} &= -\frac{\partial\Phi}{\partial p} = -f_0\frac{\partial\psi}{\partial p} \\
&\approx -\frac{f_0}{\delta p}(\psi_3-\psi_1) \\
&= \frac{f_0}{\delta p}(\psi_1-\psi_3) \\
\therefore T &\approx \left(\frac{p}{R}\right)\frac{f_0}{\delta p}(\psi_1-\psi_3)
\end{aligned}
$$

$Q_1 = -\frac{R}{p}\frac{\partial\vec{V_2}}{\partial x}\cdot\vec{\nabla}\left(\frac{p}{R}\right)\frac{f_0}{\delta p}(\psi_1-\psi_3)$ since the geostrophic wind is obtained at level 2 where $\omega \neq 0$.

$$
\begin{aligned}
\therefore Q_1 &= -\frac{f_0}{\delta p}\frac{\partial\vec{V_2}}{\partial x}\cdot\vec{\nabla}(\psi_1-\psi_3), \text{ and for} \\
Q_2 &= -\frac{f_0}{\delta p}\frac{\partial\vec{V_2}}{\partial y}\cdot\vec{\nabla}(\psi_1-\psi_3)
\end{aligned}
$$

The omega equation for the two-layer model becomes

$$
\sigma\left(\nabla^2 - 2\lambda^2\right)\omega_2 = -2\vec{\nabla}\cdot\vec{Q},
$$

if we assume that the β-effect is small enough to disregard.

Recall that

$$\psi_1 = -U_1 y + \psi_1'(x,t)$$
$$\psi_3 = -U_3 y + \psi_3'(x,t)$$
$$\omega_2 = \omega_2'(x,t)$$

$$Q_1 = -\frac{f_0}{\delta p}\left(-\frac{\partial \psi_2}{\partial y}\vec{i} + \frac{\partial \psi_2}{\partial y}\vec{j}\right)\cdot \vec{\nabla}(\psi_1 - \psi_3)$$

The perturbation streamfunction at level 2 is independent on y ($\partial\psi_2/\partial y = 0,$ and $\partial\psi_2/\partial x = \partial\psi_2'/\partial x$)

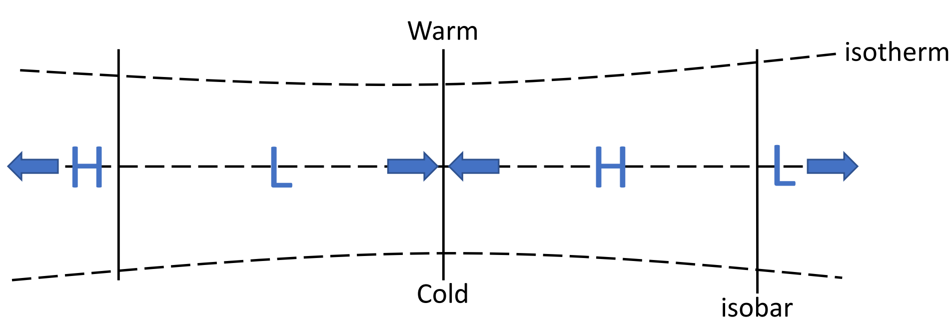
$$\begin{aligned}
Q_1 &= -\frac{f_0}{\delta p}\frac{\partial^2 \psi_2'}{\partial x^2}\vec{j}\cdot\left[\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)(\psi_1' - \psi_3')\right] \\
&= -\frac{f_0}{\delta p}\frac{\partial^2 \psi_2'}{\partial x^2}\frac{\partial}{\partial y}(\psi_1' - \psi_3') \\
&= -\frac{f_0}{\delta p}\frac{\partial^2 \psi_2'}{\partial x^2}(-U_1 + U_3) \\
&= \frac{f_0}{\delta p}\frac{\partial^2 \psi_2'}{\partial x^2}(U_1 - U_3) \\
&= \frac{2f_0}{\delta p}U_T\zeta_2' \text{ since } U_T = 1/2(U_1 - U_3) \text{ and } \zeta' = \nabla^2\psi'
\end{aligned}$$

For Q_2:

$$\begin{aligned}
Q_2 &= -\frac{f_0}{\delta p}\frac{\partial \vec{V_2}}{\partial y}\cdot\vec{\nabla}(\psi_1 - \psi_3) \\
&= -\frac{f_0}{\delta p}\frac{\partial}{\partial y}\left(\frac{\partial \psi_2'}{\partial x}\vec{j}\right)\cdot\vec{\nabla}(\psi_1 - \psi_3) \\
&= 0 \text{ since } \psi_2' \neq \psi_2'(y)
\end{aligned}$$

$$\begin{aligned}
\sigma\left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2 &= -2\vec{\nabla}\cdot\left(\frac{2f_0}{\delta p}U_T\zeta_2'\vec{i}\right) \\
\therefore \left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2 &= -\frac{4f_0}{\sigma\delta p}U_T\frac{\partial \zeta_2'}{\partial x}
\end{aligned}$$

Consider an idealised pattern at the $500\,\mathrm{hPa}$ level of isobars and isotherms as well as Q-vectors (arrows) in the Southern Hemisphere.



Recall that the Q-vector of the quasi-geostrophic model may be simplified to

$$\vec{Q} = \frac{R}{p}\frac{\partial T}{\partial y}\left(\vec{k} \times \frac{\partial \vec{V}_g}{\partial x}\right)$$

At level 2:

$$\begin{aligned}\vec{Q} &= \frac{R}{p}\frac{\partial T}{\partial y}\left(\vec{k} \times \frac{\partial}{\partial x}\left(\frac{\partial \psi_2}{\partial x}\vec{j}\right)\right)\\ \therefore Q_1 &= -\frac{R}{p}\frac{\partial T}{\partial y}\frac{\partial^2 \psi_2'}{\partial x^2}\\ &= -\frac{R}{p}\frac{\partial T}{\partial y}\zeta_2'\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{2f_0}{\delta p}U_T\zeta_2' &= -\frac{R}{p}\frac{\partial T}{\partial y}\zeta_2'\\ \therefore U_T &= -\frac{R\delta p}{2pf_0}\frac{\partial T}{\partial y}\end{aligned}$$

In the Southern Hemisphere where $f_0 < 0, U_T \propto \dfrac{\partial T}{\partial y}$, and therefore the shear of the perturbation meridional velocity tends to advect cold air equatorward west of the $500\,\mathrm{hPa}$ trough and warm air poleward east of the $500\,\mathrm{hPa}$ trough so that there is a tendency to produce a positive temperature gradient directed eastward at the trough, the same direction as the Q-vector.

We have already established that the omega equation for the two-layer model is $\sigma\left(\nabla^2 - 2\lambda^2\right)\omega_2 = -2\vec{\nabla}\cdot\vec{Q}$. This equation reduces to

$$\begin{aligned}\sigma\left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2' &= -2\vec{\nabla}\cdot\left(\frac{2f_0}{\delta p}U_T\zeta_2'\vec{i}\right)\\ \therefore \left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2' &= -\frac{4f_0}{\sigma\delta p}U_T\frac{\partial \zeta_2'}{\partial x}\end{aligned}$$

for baroclinically unstable waves.

As has been shown previously that a Laplacian of a function is the negative of that function, we have

$$\sigma\left(\frac{\partial^2}{\partial x^2} - 2\lambda^2\right)\omega_2' \propto -\omega_2' \propto w_2'$$

w_2 being the vertical component of velocity at level 2

$$\therefore \left|-\frac{4f_0}{\sigma\delta p}U_T\frac{\partial \zeta_2'}{\partial x}\right| \propto w_2'$$

For the Southern Hemisphere $f_0 < 0$, therefore, for upward motion $(w_2' > 0)$

$$\begin{aligned}-\frac{4f_0}{\sigma\delta p}U_T\frac{\partial \zeta_2'}{\partial x} &> 0\\ \therefore U_T\frac{\partial \zeta_2'}{\partial x} &> 0\end{aligned}$$

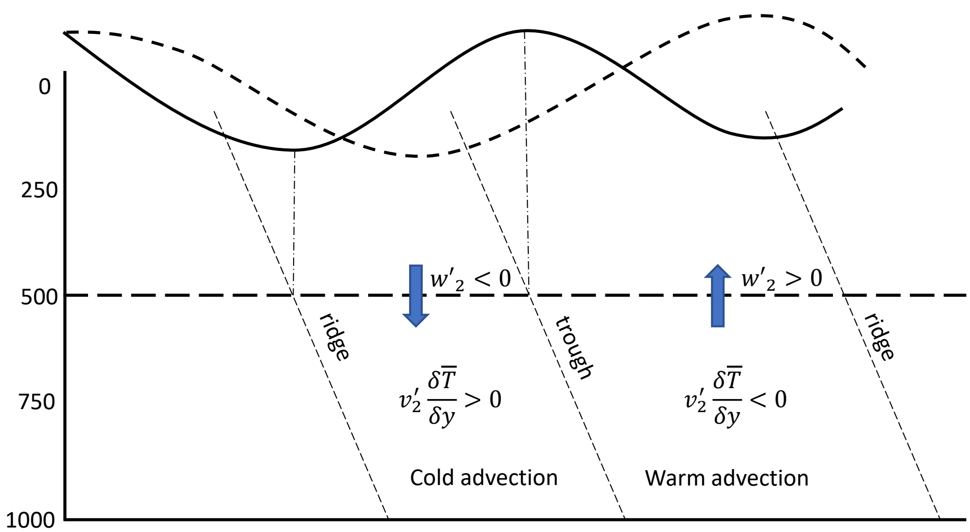
On the $500\,\mathrm{hPa}$ trough axis in the Southern Hemisphere $\zeta_2' < 0$, and at the $500\,\mathrm{hPa}$ ridge $\zeta_2' > 0$. Therefore, the advection of perturbation vorticity in the region between the trough axis and ridge axis is positive, that is $\dfrac{\partial \zeta_2'}{\partial x} > 0$. Since $U_T > 0$ (the zonal wind at $250\,\mathrm{hPa}$ is in general larger than the zonal wind at $750\,\mathrm{hPa}$)

$$U_T \frac{\partial \zeta_2'}{\partial x} > 0,$$

and this relationship is associated with $w_2' > 0$, hence upward or rising motion east of the $500\,\mathrm{hPa}$ trough. Similarly, sinking motion is found east of the $500\,\mathrm{hPa}$ ridge.

The earlier discussion on the tendency equation showed that below the $500\,\mathrm{hPa}$ ridge, ahead of the surface trough, warm advection is found so that $\vec{V}_g \cdot \vec{\nabla}\left(\dfrac{\partial \Phi}{\partial p}\right) < 0$ in the lower troposphere. For the two-layer model, the lower troposphere is below level 2 ($500\,\mathrm{hPa}$) so that the temperature advection can be written as $\vec{V}_2 \cdot \vec{\nabla}\left(\dfrac{RT}{p}\right)$, which equals $(u_2'\vec{i} + v_2'\vec{j}) \cdot \dfrac{\partial T}{\partial y}\vec{j} = v_2' \dfrac{\partial \overline{T}}{\partial y}$, where $\overline{T}$ is the average temperature of the lower troposphere. Therefore, for warm advection east of the surface trough $v_2' \dfrac{\partial \overline{T}}{\partial y} < 0$ and for cold advection east of the surface ridge $v_2' \dfrac{\partial \overline{T}}{\partial y} > 0$.

Consider an east-west section of the two-layer model that shows backward tilting trough and ridge axes with height as straight lines. For baroclinically unstable development, the ψ_1' field ($250\,\mathrm{hPa}$) lags the ψ_3' field ($750\,\mathrm{hPa}$) by about one-quarter wavelength, similar to the solutions presented earlier for a wavelength of $6000\,\mathrm{km}$.



The thickness and vertical motion fields (w_2') are in phase, and this thickness field is also in phase with the temperature advection by the perturbation meridional wind $\left(v_2' \dfrac{\partial \overline{T}}{\partial y}\right)$. Therefore, the temperature advection acts to intensify the thickness filed, acting to increase the strength of the disturbance.

Rising air in the atmosphere must be balanced by equal sinking air. Therefore, mass convergence into a given column of air must be balanced by a net mass divergence. For the two-layer model, this requirement implies that east of the ridge at $750\,\mathrm{hPa}$ divergence is a consequence of the sinking or subsiding air, and thus at $250\,\mathrm{hPa}$ convergence occurs. East of the trough, convergence happens at $750\,\mathrm{hPa}$ with divergence aloft at $250\,\mathrm{hPa}$. Next we want to establish the role divergent circulation has on how vorticity changes over time since extreme values of vorticity at the troughs and ridges act to increase the strength of the disturbance.

We have already developed a set of prediction equations in the variables ψ_1, ψ_3 and ω_2. First consider the prediction equation for ψ_1:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_1\right) = -\vec{V_1}\cdot\vec{\nabla}\left(\nabla^2\psi_1\right) - \beta\frac{\partial\psi_1}{\partial x} + \frac{f_0}{\delta p}\omega_2$$

Applying perturbation analysis and ignoring β terms leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) = -U_1\frac{\partial}{\partial x}\frac{\partial^2\psi_1'}{\partial x^2} + \frac{f_0}{\delta p}\omega_2'$$
$$\frac{\partial}{\partial t}\zeta_1' = -U_1\frac{\partial}{\partial x}\zeta_1' + \frac{f_0}{\delta p}\omega_2'$$

This equation state that the vorticity tendency at $250\,\mathrm{hPa}$ is determined by the sum of vorticity advection and the divergent circulation.

Consider the vorticity advection terms first. Between the ridge and trough axes, vorticity decreases in the x-direction in the Southern Hemisphere. Therefore,

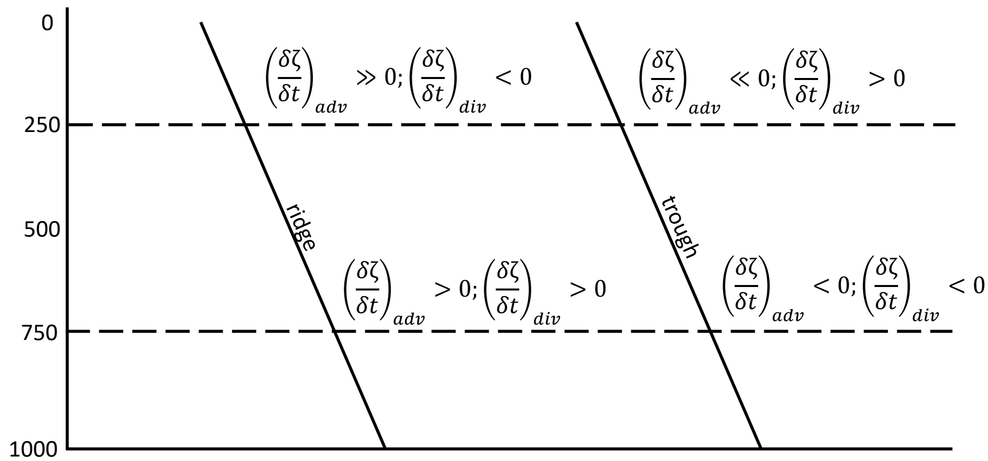
$$\frac{\partial\zeta_1'}{\partial x} < 0$$
$$\therefore -U_1\frac{\partial\zeta_1'}{\partial x} > 0 \text{ since } U_1 > 0.$$

Also, $-U_3\dfrac{\partial\zeta_3'}{\partial x} > 0$ east of the ridge axis. However, since the mean zonal wind at $250\,\mathrm{hPa}$ is much stronger than the mean zonal wind at $750\,\mathrm{hPa}$, $-U_1\dfrac{\partial\zeta_1'}{\partial x} \gg 0$ east of the ridge axis. Following similar arguments, $-U_1\dfrac{\partial\zeta_1'}{\partial x} \ll 0$ and $-U_3\dfrac{\partial\zeta_3'}{\partial x} > 0$ east of the trough axis. This change of the vorticity advection with height, i.e. differential vorticity advection, would lead to the upper level trough and ridge pattern to move faster than the lower level pattern. The result is that the westward tilt of the trough-ridge pattern will be destroyed.

The divergent circulation part of the prediction equations at level 1 ($250\,\mathrm{hPa}$) is $\dfrac{f_0}{\delta p}\omega_2' = -\dfrac{f_0}{\delta p}w_2'$ and $-\dfrac{f_0}{\delta p}\omega_2' = \dfrac{f_0}{\delta p}w_2'$ at level 3 ($750\,\mathrm{hPa}$).

In the area of sinking motion, east of the ridge, $w_2' < 0$, and in the area of rising motion, east of the trough, $w_2' > 0$. Therefore, east of the ridge at level 1, $-\dfrac{f_0}{\delta p}\omega_2' < 0$ in the Southern Hemisphere where $f_0 < 0$. Also in the Southern Hemisphere, east of the ridge at level 3, $\dfrac{f_0}{\delta p}\omega_2' > 0$; east of the trough at level 1, $-\dfrac{f_0}{\delta p}\omega_2' > 0$; east of the trough at level 3, $\dfrac{f_0}{\delta p}\omega_2' < 0$.

The schematic below shows the vertical cross section for an unstable baroclinic wave in the two-layer model that shows the different phases of vorticity change $\left(\frac{\delta\zeta}{\delta t}\right)$.



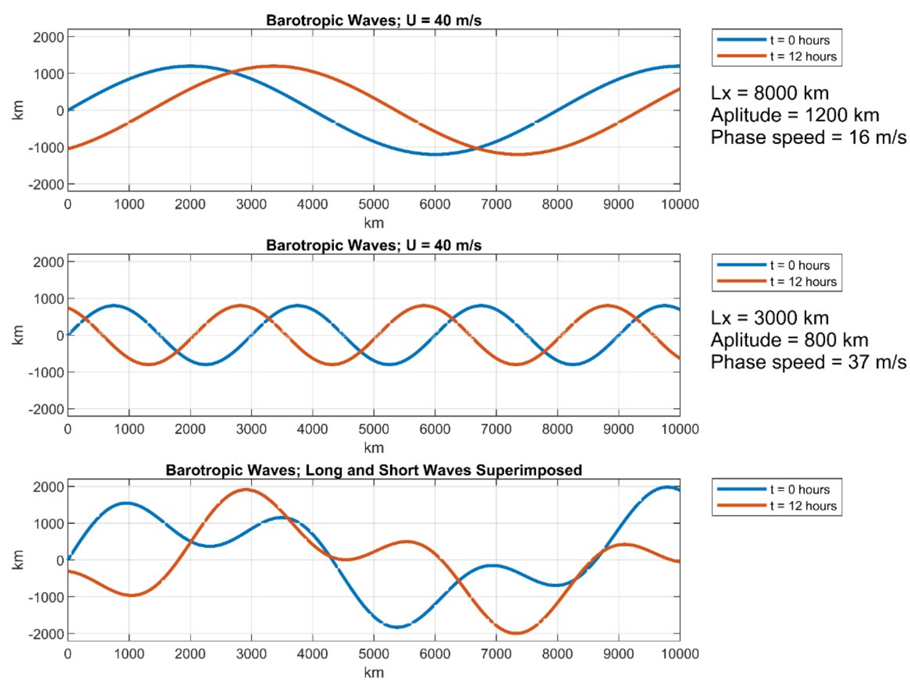
From the schematic it is clear that the contributions from the vorticity advection and the divergent circulation at the lower level are working in unison, while at the upper level the two contributions oppose each other. The result of this configuration is that the westward tilt of the trough-ridge pattern is maintained. The maintenance of the tilt is thus due to the divergent secondary circulation. Moreover, this secondary circulation tends to amplify the vorticity perturbations in the troughs and ridges at both levels, resulting in a further growing of the disturbance.

Examples of theoretical barotropic and baroclinic waves

Next we further investigate Rossby waves properties by approximating them by simple sine and cosine functions. In order to see the effect of phase speed and wave amplitude on barotropic Rossby waves, the following equation is used to describe a meandering jet stream around a reference latitude of $50°$S.

$$y' = A \sin \left[2\pi \left(\frac{x - ct}{L_x} \right) \right],$$

that describes the displacement distance of the Rossby wave streamlines around a reference latitude, in this case $50°$S. We consider winds of $40\,\mathrm{m/s}$ in the jet and consider, respectively, long and short waves. The former has a wave length (L_x) of $8000\,\mathrm{km}$ and wave amplitude of $1200\,\mathrm{km}$, while the latter's wavelength is $3000\,\mathrm{km}$ and its amplitude is $800\,\mathrm{km}$.



The wavenumber, k, is equal to $\dfrac{2\pi}{L_x}$, and the barotropic phase speed, c, is equal to $40 - \dfrac{\beta}{k^2}$ with $\beta = 1.47 \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$. The figure below shows the results for the two waves as well as the positioning of these waves 12 hours later. Also shown in the bottom panel is the result of superimposing the long and short waves. The short waves move along the long wave streamline. Such fast moving short waves cause rapid changes in the weather, and weather forecasters need to be paying particular attention to them.

To mimic north-south displacement for a baroclinic wave, we have

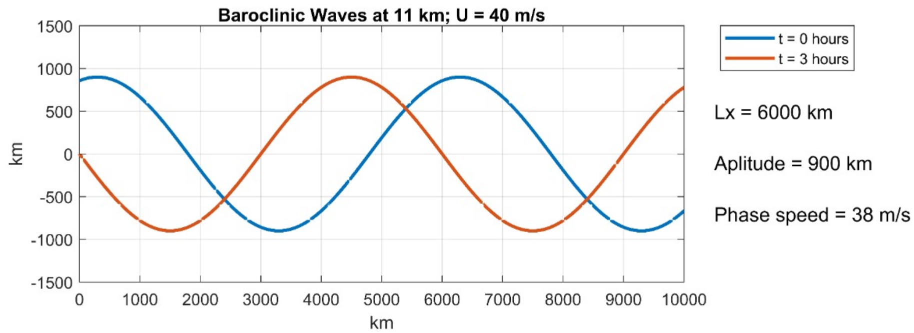
$$y' = B \cos\left(\frac{\pi z}{z}\right) \sin\left[2\pi \left(\frac{x - ct}{L_x}\right)\right],$$

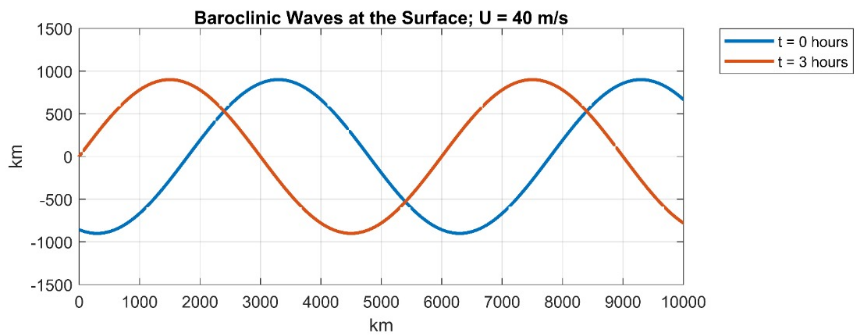
and take into consideration that the phase speed, c, is equal to $40 - \beta\left(k^2 + 2\lambda^2\right)^{-1}$. Recall that $\lambda^2 = \dfrac{f_0^2}{\sigma \delta p^2}$, which is a reminder that static stability, σ, has an effect on baroclinic waves. Cold strongly stable air near the poles restricts vertical movement of air, while warm, weakly stable air near the equator is less limiting in the vertical. Notwithstanding these effects, we consider average mid-latitude tropospheric conditions with $\lambda^2 \simeq 2 \times 10^{-12}\,\mathrm{m}^{-2}$.

Take note that there is an additional cosine factor introduced into the barotropic Rossby wave stream flow function

$$B \cos\left(\frac{\pi z}{z}\right),$$

with $z = 11\,\mathrm{km}$ (the tropospheric depth) and B, the wave amplitude, equal to $900\,\mathrm{km}$. Here we used $6000\,\mathrm{km}$ as the wavelength. The cosine factor causes the meridional wave amplitude to be equal to 1 at the surface (since $\zeta = 0$ and $\cos(0) = 1$), and -1 at the top of the troposphere (since $\zeta = 11\,\mathrm{km}$ and $\cos(\pi) = -1$). The two resulting waves are subsequently $180°$ out of phase between top and bottom of the troposphere. Although this out of phase result is an oversimplification, the figure gives some insight into how baroclinic waves function. As with the barotropic wave example, waves are presented at an initial time and at a later time. Here we show results after 3 hours. Clearly, one can see that an upper level trough (ridge) is above a surface ridge (trough).


Baroclinic Waves at 11 km; U = 40 m/s
t = 0 hours
t = 3 hours
Lx = 6000 km
Aplitude = 900 km
Phase speed = 38 m/s
km
km


Baroclinic Waves at the Surface; U = 40 m/s
t = 0 hours
t = 3 hours
km
km

The energetics of baroclinic waves

In baroclinic instability, warm air rises and moves poleward, while cold air sinks and moves equatorward. This motion extracts available potential energy from the mean flow. As the instability grows exponentially, the energy of the perturbation is not conserved, and both the kinetic energy and available potential energy of the perturbation increase. Subsequently, it is necessary to determine the role of the potential energy (and its conversion to kinetic energy) of the mean flow as an energy source.

Recall the set of perturbation equation derived before:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_1'}{\partial x^2}+\beta\frac{\partial\psi_1'}{\partial x}&=\frac{f_0}{\delta p}\omega_2'\\ \left(\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_3'}{\partial x^2}+\beta\frac{\partial\psi_3'}{\partial x}&=-\frac{f_0}{\delta p}\omega_2'\\ \left(\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right)\left(\psi_1'-\psi_3'\right)-U_T\frac{\partial}{\partial x}\left(\psi_1'+\psi_3'\right)&=\frac{\sigma\delta p}{f_0}\omega_2'.\end{aligned}$$

We use these equations to derive the energy equations for the system.

The first prediction equation is multiplied by $-\psi_1'$ to produce

$$-\psi_1'\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)-U_1\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)-\psi_1'\beta\frac{\partial\psi_1'}{\partial x}=-\psi_1'\frac{f_0}{\delta p}\omega_2'.$$

In order to manage the first term on the left, we apply the product rule

$$\begin{aligned}\frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial t}\right)\right)&=\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial t}\right)\right)+\frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial t}\right)\\ &=\psi_1'\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)+\frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial\psi_1'}{\partial x}\right)\\ \therefore -\psi_1'\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)&=-\frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial t}\right)\right)+\frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial\psi_1'}{\partial x}\right)\end{aligned}$$

Integrate the resulting prediction equation over one wavelength of the perturbation in the zonal direction. The zonally averaged terms, $\overline{(\ )}$, are presented by

$$\frac{1}{L}\int_0^L(\ )dx,$$

where L is the wavelength of the perturbation.

The integral of the prediction equation in ψ_1' is then

$$\frac{1}{L}\int_0^L \left[-\frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial \psi_1'}{\partial t}\right)\right)+\frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)-U_1\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_1'}{\partial x^2}\right)-\psi_1'\beta\frac{\partial \psi_1'}{\partial x}+\psi_1'\frac{f_0}{\delta p}\omega_2'\right]dx=0$$

Consider the first term of the left

$$-\frac{1}{L}\int_0^L \frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial \psi_1'}{\partial t}\right)\right)dx$$

This term is equal to zero because of the rule of integrals of a perfect differential. To better understand this rule, consider a scalar function in the zonal direction, $Q = Q(x)$

$$dQ \equiv \frac{\partial Q}{\partial x}dx$$

for the zonal case.

The perfect differential for a differentiable function Q is equal to $dQ = \vec{\nabla}Q\cdot d\vec{r}$ with $d\vec{r} = dx$ in our case of zonal flow, and $\vec{\nabla}Q$ is the gradient of Q so that the gradient theorem states

$$\int_i^f dQ = \int_i^f \vec{\nabla}Q\cdot d\vec{r} = Q(f) - Q(i)$$

where i and f represent endpoints of an integral path. This result means that the integral of a perfect differential is independent of the integration path. Owing to this integral path independence, $\vec{\nabla}\times\vec{\nabla}Q = 0$, and according to Stoke's theorem

$$\oint \vec{\nabla}Q\cdot d\vec{r} = \iint \vec{\nabla}\times\vec{\nabla}Q\cdot d\vec{a} = 0.$$

$$\therefore \frac{1}{L}\int_0^L \left[\underbrace{\frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)}_{\text{A}}\underbrace{-U_1\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_1'}{\partial x^2}\right)}_{\text{B}}\underbrace{-\psi_1'\beta\frac{\partial \psi_1'}{\partial x}}_{\text{C}}\underbrace{+\psi_1'\frac{f_0}{\delta p}\omega_2'}_{\text{D}}\right]dx=0$$

To evaluate Term A, first consider $\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)^2$, which is equal to

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial \psi_1'}{\partial x}\right)\left(\frac{\partial \psi_1'}{\partial x}\right)\right]\\ =&\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)\frac{\partial \psi_1'}{\partial x}+\frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)\right]\\ =&\frac{1}{2}\left[2\frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)\right]\end{aligned}$$

Therefore, Term A, $\overline{\frac{\partial \psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial \psi_1'}{\partial x}\right)}$, can be written as $\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial \psi_1'}{\partial x}\right)^2}$. This term represents the rate of change of the perturbation kinetic energy per unit mass averaged over a wavelength.

To evaluate Term B, consider the following expansion

$$\frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)\right) = \frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right) + \psi_1'\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi_1'}{\partial x}\right)$$

$$\therefore \psi_1'\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi_1'}{\partial x}\right) = \frac{\partial}{\partial x}\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)\right) - \frac{\partial\psi_1'}{\partial x}\frac{\partial^2\psi_1'}{\partial x^2},$$

then evaluate the second term on the right by considering

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)^2 &= \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\psi_1'}{\partial x}\right)\left(\frac{\partial\psi_1'}{\partial x}\right)\right] \\ &= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)\frac{\partial\psi_1'}{\partial x} + \frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)\right] \\ &= \frac{\partial\psi_1'}{\partial x}\frac{\partial^2\psi_1'}{\partial x^2}\end{aligned}$$

Term B then becomes

$$-U_1\overline{\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)} = -U_1\frac{\partial}{\partial x}\overline{\left(\psi_1'\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)\right)} + \frac{U_1}{2}\frac{\partial}{\partial x}\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2}$$

Both of the terms on the right are perfect differentials and therefore vanish. However, take note that the term $\frac{U_1}{2}\frac{\partial}{\partial x}\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2}$ is also a kinetic energy term and is also associated with the advection term. Therefore, the advection of kinetic energy vanishes when integrated over a wavelength.

For Term C, it is easily seen that $-\beta\psi_1'\frac{\partial\psi_1'}{\partial x} = -\frac{\beta}{2}\frac{\partial}{\partial x}(\psi_1')^2$, therefore, $-\beta\overline{\psi_1'\frac{\partial\psi_1'}{\partial x}} = -\frac{\beta}{2}\frac{\partial}{\partial x}\overline{(\psi_1')^2} = 0$. Since Term D cannot be further simplified, the prediction equation at level 1, that we can apprehend as a permutation energy equation is thus

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2} + \frac{f_0}{\delta p}\overline{\omega_2'\psi_1'} = 0.$$

Similarly, at level 3

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial\psi_3'}{\partial x}\right)^2} - \frac{f_0}{\delta p}\overline{\omega_2'\psi_3'} = 0.$$

Next, we consider the prediction equation in terms of U_m and U_T by multiplying the equation through by $(\psi_1' - \psi_3')$ and integrating over one wavelength.

Therefore,

$$\frac{1}{L}\int_0^L\left[\underbrace{(\psi_1'-\psi_3')\frac{\partial}{\partial t}(\psi_1'-\psi_3')}_{\text{A}} + \underbrace{U_m(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'-\psi_3')}_{\text{B}} \underbrace{-U_T(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'+\psi_3')}_{\text{C}} \underbrace{-(\psi_1'-\psi_3')\frac{\sigma\delta p}{f_0}\omega_2'}_{\text{D}}\right]dx = 0.$$

From Term A, integrating $(\psi_1'-\psi_3')\frac{\partial}{\partial t}(\psi_1'-\psi_3')$ produces, as before, an energy term

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{(\psi_1'-\psi_3')^2}.$$

Similarly, Term B leads to

$$\frac{U_m}{2}\frac{\partial}{\partial x}\overline{(\psi_1'-\psi_3')^2},$$

which vanishes since it is a perfect differential. Terms C and D cannot be simplified further. Therefore, we get the third energy perturbation equation

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{(\psi_1'-\psi_3')^2}-U_T\overline{(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'+\psi_3')}-\frac{\sigma\delta p}{f_0}\overline{(\psi_1'-\psi_3')\omega_2'}=0$$

The total perturbation kinetic energy is defined as

$$\begin{aligned}
K' &\equiv \frac{1}{2}\left[\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2}+\overline{\left(\frac{\partial\psi_3'}{\partial x}\right)^2}\right]\\
\therefore \frac{dK'}{dt} &= \frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2}+\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(\frac{\partial\psi_3'}{\partial x}\right)^2}\\
&= -\frac{f_0}{\delta p}\overline{\omega_2'\psi_1'}+\frac{f_0}{\delta p}\overline{\omega_2'\psi_3'}\\
&= -\frac{f_0}{\delta p}\overline{\omega_2'(\psi_1'-\psi_3')}\\
&= -\frac{2f_0}{\delta p}\overline{\omega_2'\psi_T} > 0 \text{ for an increase in perturbation kinetic energy.}
\end{aligned}$$

ψ_T is the perturbation thickness for baroclinic flow. Therefore, the rate of change of perturbation kinetic energy is proportional to the interrelationship between perturbation thickness and vertical motion.

Now we define the perturbation available potential energy as

$$\begin{aligned}
P' &\equiv \frac{\lambda^2}{2}\overline{(\psi_1'-\psi_3')^2} \text{ with } \lambda^2=\frac{f_0^2}{\sigma\delta p^2}\\
\therefore \frac{dP'}{dt} &= \frac{\lambda^2}{2}\frac{\partial}{\partial t}\overline{(\psi_1'-\psi_3')^2}\\
&= \lambda^2U_T\overline{(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'+\psi_3')}+\frac{f_0^2}{\sigma\delta p^2}\frac{\sigma\delta p}{f_0}\overline{(\psi_1'-\psi_3')\omega_2'}\\
&= \lambda^2U_T\overline{(2\psi_T)\frac{\partial}{\partial x}(2\psi_m)}+\frac{f_0}{\delta p}\overline{(2\psi_T)\omega_2'}\\
&= 4\lambda^2U_T\overline{\psi_T\frac{\partial}{\partial x}\psi_m}+\frac{2f_0}{\delta p}\overline{\omega_2'\psi_T}\\
\therefore \frac{dP'}{dt} &= 4\lambda^2U_T\overline{\psi_T\frac{\partial}{\partial x}\psi_m}-\frac{dK'}{dt}
\end{aligned}$$

The term $\frac{2f_0}{\delta p}\overline{\omega_2'\psi_T}$ must, therefore, represent a conversion between perturbation potential and perturbation kinetic energy. To further investigate the interrelationship between the vertical motion and thickness variables, four cases are considered.

The four cases are:

1. The vertical motion is upward and the thickness is greater than average.
2. The vertical motion is upward and the thickness is less than average.
3. The vertical motion is downward and the thickness is greater than average.
4. The vertical motion is downward and the thickness is less than average.

Take note that in the Northern Hemisphere where $\psi_1' > 0$ (since both geopotential and f_0 are positive), thickness is greater then average when $(\psi_1' - \psi_3')_{\text{NH}} > 0$, and thickness is less than average when $(\psi_1' - \psi_3')_{\text{NH}} < 0$. Therefore, in the northern Hemisphere case for thickness to be greater than average when $\psi_1' > \psi_3'$ and for thickness in the Northern Hemisphere to be less than average $\psi_1' < \psi_3'$.

Recall that

$$\psi_1 = -U_1 y + \psi_1'$$
$$\therefore \psi_1' = \psi_1 + U_1 y = \frac{\Phi_1}{f_0} + U_1 y.$$

Since $f_0 < 0$ and $y < 0$ in the Southern Hemisphere, $\psi_1' < 0$. Similarly, $\psi_3' < 0$ in the Southern Hemisphere. As a result for thickness greater than average $\psi_1' < \psi_3'$ and thus $\psi_1' - \psi_3' < 0$. Therefore $\psi_T < 0$ when thickness is greater than average in the Southern Hemisphere. For thickness less than average in the Southern Hemisphere $\psi_1' > \psi_3'$ and thus $\psi_1' - \psi_3' > 0$. Therefore $\psi_T > 0$ when thickness is less than average in the Southern Hemisphere.

We can now evaluate the four cases for the Southern Hemisphere:

$$\text{Case 1. } \frac{dK'}{dt} \sim -(f_0 < 0)(\omega_2' < 0)(\psi_T < 0) > 0$$
$$\text{Case 2. } \frac{dK'}{dt} \sim -(f_0 < 0)(\omega_2' < 0)(\psi_T > 0) < 0$$
$$\text{Case 3. } \frac{dK'}{dt} \sim -(f_0 < 0)(\omega_2' > 0)(\psi_T < 0) < 0$$
$$\text{Case 4. } \frac{dK'}{dt} \sim -(f_0 < 0)(\omega_2' > 0)(\psi_T > 0) > 0$$

Therefore, perturbation potential energy is being converted to kinetic energy when on average the vertical motion is positive $(\omega_2' < 0)$ where thickness is greater than average ($\psi_1' - \psi_3' < 0$ in the Southern Hemisphere), and vertical motion is negative $(\omega_2' > 0)$ where thickness is less than average ($\psi_1' - \psi_3' > 0$ in the Southern Hemisphere).

The available potential energy and kinetic energy of a disturbance are able to grow simultaneously on condition that

$$4\lambda^2 U_T \overline{\psi_T \frac{\partial \psi_m}{\partial x}} > \frac{2f_0}{\delta p}\overline{\omega_2' \psi_T},$$

which means that the potential energy generation needs to exceed the rate of potential energy conversion to kinetic energy.

The potential energy generation term on the left hand side depends on the interrelationship between the perturbation thickness (ψ_T) and the meridional velocity ($\partial \psi_m / \partial x$, the change of the baroclinic perturbation in the zonal direction) at $500\,\mathrm{hPa}$. If we suppose the barotropic and baroclinic parts, this disturbance can be represented as such

$$\psi_m = A_m \cos k(x - ct) \text{ and}$$
$$\psi_T = A_T \cos k(x + x_0 - ct),$$

with the role of the x_0 term to indicate the phase difference, we can investigate the relative positions of the temperature and geopotential waves of the baroclinic disturbance. Take note that A_m and A_T represent amplitudes of respectively the $500\,\mathrm{hPa}$ disturbance geopotential and thickness fields.

Appendix A explains how the following relationship is obtained

$$\overline{\psi_T \frac{\partial \psi_m}{\partial x}} = \frac{1}{2} A_T A_m k \sin(kx_0).$$

For the perturbation potential energy to increase, we thus have

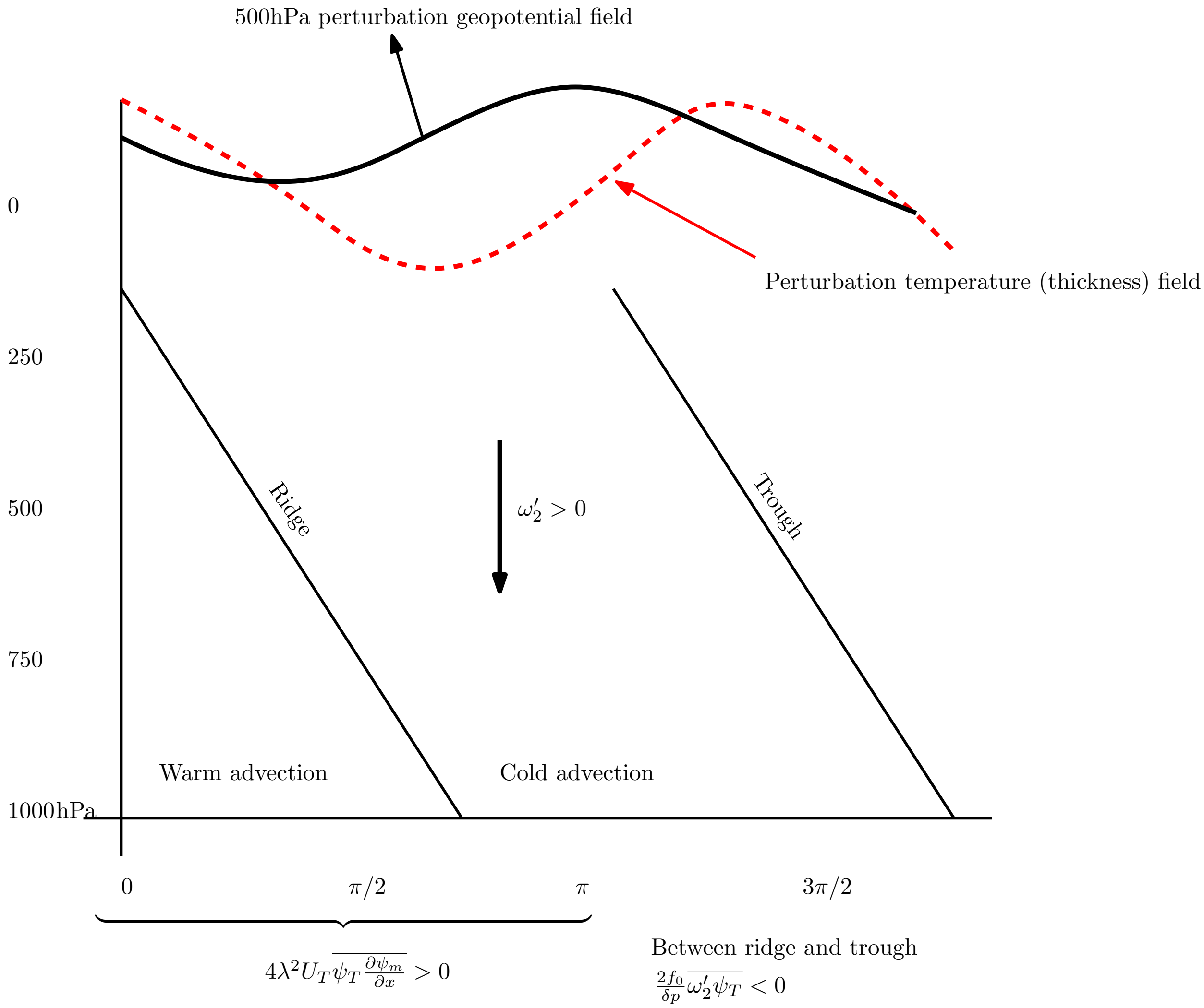
$$2\lambda^2 A_T A_m U_T k \sin(kx_0) + \frac{2f_0}{\delta p}\overline{\omega_2' \psi_T} > 0.$$

Since we have already discussed under which conditions the energy conversion term is positive (i.e. for cases 1 and 4), $2\lambda^2 A_T A_m U_T k \sin(kx_0)$ still needs to be evaluated. For the case of a mid-latitude westerly thermal wind ($U_T > 0$), $\sin(kx_0)$ must, therefore, be positive, and thus $kx_0 > 0$. Moreover, for $\sin(kx_0)$ to remain positive during one sinusoidal cycle, $kx_0 < \pi$. Therefore, $0 < kx_0 < \pi$. Take note that the interrelationship, represented by the sinusoidal function has a maximum value at $kx_0 = \pi/2$ (i.e. $90°$) which means that the phase of the perturbation thickness (temperature) wave lags the meridional velocity (geopotential) wave by $90°$ at the $500\,\mathrm{hPa}$ level, i.e. a one-quarter cycle.

The phase shift description is shown schematically in the east-west section of the z-layer model that includes the backward tilting of the trough and ridge axes shown below. In the ideal configuration of a one-quarter cycle phase shift, the temperature wave lags the geopotential wave, resulting in a southward advection of warm air by the geostrophic wind east of the $500\,\mathrm{hPa}$ trough, and a northward advection of cold air west of the $500\,\mathrm{hPa}$ trough in the Southern Hemisphere. Both of these advections are maximised as a result of the $90°$ phase shift, since $\sin(kx_0)$ has a positive maximum value at that phase. This maximisation results in the cold advection to be strong below the $250\,\mathrm{hPa}$ trough and the warm advection to be strong below the $250\,\mathrm{hPa}$ ridge. Therefore, based on our understanding of the geopotential tendency equation, the upper level disturbance with westward tilting trough and ridge axes, will intensify.

East of the ridge, behind the trough, downward motion ($\omega_2' > 0$) and cold advection occurs. The latter is associated with the thickness being less than average, which means that in the Southern Hemisphere $\psi_T > 0$ there. Therefore, $\overline{\omega_2' \psi_T} > 0$, and $\frac{2f_0}{\delta p}\overline{\omega_2' \psi_T} < 0$ in the Southern Hemisphere ($f_0 < 0$), which is the second term of the equation describing the time rate of change of the perturbation available potential energy. The first term of this equation is positive between the phases of zero and π. As a result, in this westward tilting configuration we see that the sign of the two terms differs there.

We can therefore conclude that horizontal temperature advection $\left(\overline{\psi_T \frac{\partial \psi_m}{\partial x}}\right)$ increases available potential energy of the perturbation, while the vertical circulation $\left(\overline{\omega_2' \psi_T}\right)$ converts perturbation available energy to perturbation kinetic energy since $\frac{dK'}{dt}$ equals the negative of the $\overline{\omega_2' \psi_T}$ term, resulting in $\frac{dK'}{dt} > 0$.



Take note that it is only the potential energy generation term that determines the growth of the total energy since

$$\frac{d}{dt}(P' + K') = 4\lambda^2 U_T \overline{\psi_T \frac{\partial \psi_m}{\partial x}}.$$

We therefore see that with $U_T > 0$ and with the positive interrelationship between temperature and the meridional velocity, the total energy of the perturbation increases. The conversion of the available potential energy to kinetic energy is brought about by the vertical circulation and this conversion does not affect the total energy of the perturbation.

The total energy of the perturbation is not conserved since both the potential and kinetic energy are increasing as they extract energy from the mean state. As can be seen from the diagram, the vertical velocity is in phase with the temperature field. However, if the vertical velocity becomes 90° out of phase with the temperature in the life cycle of the disturbance, there is no longer a conversion of perturbation available potential energy to perturbation kinetic energy and the disturbance is restored to a mean state.

The global energy cycle

We showed that there is a positive interrelationship between vertical motion and temperature anomalies that leads to the generation of kinetic energy of atmospheric flow. We split the energy of the flow (kinetic and available potential) into energy associated with the mean zonal flow and into energy associated with the eddies (i.e., deviations from the zonal mean). It is useful to examine the exchange of energy between the eddies and the mean flow, but before we can do this we first develop the so-called Eulerian mean equations in log-pressure coordinates. As up to this point in the book, the primary emphasis is on extra-tropical aspects of the circulation.

The transformation from standard isobaric coordinates to a vertical coordinate based on the logarithm of pressure, we begin by defining the vertical coordinate z^* as $-H \ln\left(\dfrac{p}{p_s}\right)$ with H as a standard scale height and p_s a standard reference pressure. The horizontal momentum equation is the same as that of the isobaric system

$$\frac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} + \vec{\nabla}\Phi = 0,$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} + w^* \frac{\partial}{\partial z^*}$$

w^* is the vertical velocity in the log-pressure coordinate system with $w^* \equiv \dfrac{Dz^*}{Dt}$. Here, z^* is considered to be exactly equal to geometric height and the density, $\rho_0 = \rho_0(z^*)$. Next we derived the hydrostatic, continuity and thermodynamic equations for the log-pressure coordinate system.

The transformation of the hydrostatic equation, $\dfrac{\partial \Phi}{\partial p} = -\dfrac{RT}{p}$, is done in the following way. We get

$$\begin{aligned} p\frac{\partial \Phi}{\partial p} &= -RT \\ \therefore -H\frac{\partial \Phi}{\partial \ln p} &= \frac{RT}{H} \\ \therefore \frac{\partial \Phi}{\partial z^*} &= \frac{RT}{H} \text{ when using the definition of } z^* \end{aligned}$$

To obtain the log-pressure form of the continuity equation we consider this equation in the isobaric system as $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$, and define the vertical velocity $w^* \equiv \left(-\frac{H}{p}\right)\frac{Dp}{Dt}=-\frac{H}{p}\omega$, therefore $\omega=-\frac{pw^*}{H}$.

$$\begin{aligned}
\frac{\partial \omega}{\partial p} = \frac{\partial}{\partial p}\left(-\frac{pw^*}{H}\right) &= -\frac{1}{H}\left[p\frac{\partial w^*}{\partial p}+w^*\frac{\partial p}{\partial p}\right] \\
&= -\frac{\partial w^*}{H\partial \ln p}-\frac{w^*}{H} \\
&= \frac{\partial w^*}{\partial z^*}-\frac{w^*}{H}
\end{aligned}$$

Consider $\frac{\partial}{\partial z^*}(\rho_0 w^*) = w^*\frac{\partial \rho_0}{\partial z^*}+\rho_0\frac{\partial w^*}{\partial z^*}$, and define the density profile of $rho_0(z^*)$ as $\rho_s \exp\left(-\frac{z^*}{H}\right)$ with ρ_s the density at $z^*=0$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial z^*}(\rho_0 w^*) &= w^*\left(-\frac{\rho_s}{H}\exp\left(-\frac{z^*}{H}\right)\right)+\rho_0\frac{\partial w^*}{\partial z^*} \\
&= -\frac{w^*}{H}\rho_0+\rho_0\frac{\partial w^*}{\partial z^*} \\
\therefore \frac{1}{\rho_0}\frac{\partial}{\partial z^*}(\rho_0 w^*) &= \frac{\partial w^*}{\partial z^*}-\frac{w^*}{H}=\frac{\partial \omega}{\partial p}
\end{aligned}$$

The continuity equation in log-pressure coordinates then becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{\rho_0}\frac{\partial}{\partial z^*}(\rho_0 w^*)=0.$$

Next we obtain the thermodynamic energy equation in log-pressure coordinates by using the hydrostatic equation $\frac{\partial \Phi}{\partial z^*}=\frac{RT}{H}$.

We then get to replace T by $\frac{H}{R}\frac{\partial \Phi}{\partial z^*}$ in the thermodynamic energy equation

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}\right)\frac{H}{R}\frac{\partial \Phi}{\partial z^*}-S_p\omega &= \frac{J}{c_p} \\
\therefore \left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}\right)\frac{\partial \Phi}{\partial z^*}-S_p\omega\frac{R}{H} &= \frac{J\kappa}{H} \quad \text{with } \kappa=\frac{R}{c_p}
\end{aligned}$$

Consider the definition of the static stability parameter,

$$S_p \equiv \frac{RT}{c_p p}-\frac{\partial T}{\partial p}$$

$$
\begin{aligned}
\therefore -s_p\omega\frac{R}{H} &= -\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right)\left(-\frac{pw^*}{H}\right)\frac{R}{H} \\
&= \frac{R}{H}\left(\frac{\kappa T}{H} - \frac{p}{H}\frac{\partial T}{\partial p}\right) w^* \\
&= \frac{R}{H}\left(\frac{\kappa T}{H} - \frac{\partial T}{H\partial \ln p}\right) w^* \\
&= \frac{R}{H}\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right) w^*
\end{aligned}
$$

$$
\therefore \left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)\frac{\partial \Phi}{\partial z^*} + N^2 w^* = \frac{J\kappa}{H},
$$

with $N^2 \equiv \frac{R}{H}\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right)$, a measure of the static stability of the environment.

Form now on for the rest of this discussion, for convenience, the asterisk is dropped, but z still designates the log-pressure vertical coordinate, and w the vertical velocity in this coordinate system.

Consider the momentum equation, but this time include the frictional force, $\vec{F_r}$:

$$
\frac{D\vec{V}}{Dt} + f\vec{k}\times\vec{V} + \vec{\nabla}\Phi = \vec{F_r}
$$

$$
\begin{aligned}
\therefore \frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} &= X, \text{ and} \\
\frac{Dv}{Dt} + fv + \frac{\partial \Phi}{\partial x} &= Y,
\end{aligned}
$$

with X and Y representing the zonal and meridional components of drag owing to small-scale eddies.

Since $\frac{\partial \Phi}{\partial z} = \frac{RT}{H}$, we can write the thermodynamic equation as

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)\frac{RT}{H} + \frac{R}{H}\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right) w &= \frac{JR}{c_p H} \\
\therefore \left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla} + w\frac{\partial}{\partial z}\right) T + \frac{\kappa T}{H} w &= \frac{J}{c_p} \\
\therefore \frac{DT}{Dt} + \frac{\kappa T}{H} w &= \frac{J}{c_p},
\end{aligned}
$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla} + w\frac{\partial}{\partial z}$.

The continuity equation becomes

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0
$$

The zonally averaged circulation is analysed here by studying the interaction of eddies with the mean flow. The eddies are longitudinally varying disturbances and the mean flow is longitudinally averaged flow.

A variable A is expanded as $A = \overline{A} + A'$, with $\overline{A}$ the mean flow and A' the eddies. Next we create Eulerian mean equations by taking zonal averages of the equations in the log-pressure coordinate system by expanding the material derivative for variable A in flux form. See Appendix B how the following form is obtained:

$$\rho_0 \frac{DA}{Dt} = \rho_0 \left(\frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} + w \frac{\partial}{\partial z} \right) A + A \left[\vec{\nabla} \cdot (\rho_0 \vec{V}) + \frac{\partial}{\partial z}(\rho_0 w) \right]$$

Expanding the RHS of this equation leads to

$$\rho_0 \left(\frac{\partial}{\partial t} + (u\vec{i} + v\vec{j}) \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \right) \right) A + A \left[\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \right) \cdot (\rho_0 u\vec{i} + \rho_0 v\vec{j}) + \frac{\partial}{\partial z}(\rho_0 w) \right]$$
$$= \rho_0 \frac{\partial A}{\partial t} + \rho_0 \left(u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} \right) + A\frac{\partial}{\partial x}(\rho_0 u) + A\frac{\partial}{\partial y}(\rho_0 v) + A\frac{\partial}{\partial z}(\rho_0 w)$$

Since $\rho_0 \neq \rho_0(x, y, t)$ and $A \neq A(z)$, the RHS becomes

$$\frac{\partial}{\partial t}(\rho_0 A) + u\frac{\partial}{\partial x}(\rho_0 A) + v\frac{\partial}{\partial y}(\rho_0 A) + A\frac{\partial}{\partial x}(\rho_0 u) + A\frac{\partial}{\partial y}(\rho_0 v) + \frac{\partial}{\partial z}(\rho_0 A w)$$
$$= \frac{\partial}{\partial t}(\rho_0 A) + \frac{\partial}{\partial x}(\rho_0 A u) + \frac{\partial}{\partial y}(\rho_0 A v) + \frac{\partial}{\partial z}(\rho_0 A w)$$

Consider the product of variables, e.g. Au next, and applying zonal averaging:

$$\begin{aligned} \overline{Au} &= \overline{(\overline{A} + A')(\overline{u} + u')} \\ &= \overline{\overline{A}\overline{u}} + \overline{\overline{A}u'} + \overline{A'\overline{u}} + \overline{A'u'} \\ &= \overline{A}\overline{u} + \overline{A}\overline{u'} + \overline{A'}\overline{u} + \overline{A'u'} \end{aligned}$$

However, $\overline{A'} = \overline{u'} = 0$.

$$\therefore \overline{Au} = \overline{A}\overline{u} + \overline{A'u'} \text{ and } \overline{Aw} = \overline{A}\overline{w} + \overline{A'w'}.$$

Furthermore, $\dfrac{\partial \overline{()}}{\partial x} = 0$ so that

$$\rho_0 \overline{\frac{DA}{Dt}} = \frac{\partial}{\partial t}\left(\rho_0 \overline{A}\right) + \frac{\partial}{\partial y}\left(\rho_0 \left(\overline{A}\overline{v} + \overline{A'v'}\right)\right) + \frac{\partial}{\partial z}\left(\rho_0 \left(\overline{A}\overline{w} + \overline{A'w'}\right)\right).$$

Next consider the continuity equation and use zonal means again:

$$\begin{aligned} \frac{\partial}{\partial x}\overline{(\overline{u} + u')} + \frac{\partial}{\partial y}\overline{(\overline{v} + v')} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0(\overline{\overline{w} + w'})) &= 0 \\ \therefore 0 + \frac{\partial}{\partial y}(\overline{v} + \overline{v'}) + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0(\overline{v} + \overline{v'})) &= 0 \\ \therefore \frac{\partial}{\partial y}\overline{v} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \overline{w}) &= 0 \end{aligned}$$

Apply the chain rule of differentiation to the right hand side of the $\rho_0 \overline{\frac{DA}{Dt}}$ equation:

$$\frac{\partial}{\partial t}\left(\rho_0 \overline{A}\right) + \rho_0\left(\overline{A}\frac{\partial \overline{v}}{\partial y} + \overline{v}\frac{\partial \overline{A}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\rho_0 \overline{A'v'}\right) + \rho\left(A\frac{\partial \overline{w}}{\partial z} + \overline{w}\frac{\partial \overline{A}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho_0 \overline{A'w'}\right)$$
$$= \frac{\partial}{\partial t}\left(\rho_0 \overline{A}\right) + \rho_0 \overline{v}\frac{\partial \overline{A}}{\partial y} + \rho_0 \overline{w}\frac{\partial \overline{A}}{\partial z} + \rho_0 \overline{A}\left[-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \overline{w}\right)\right] + \rho_0 A\frac{\partial \overline{w}}{\partial z} + \frac{\partial}{\partial y}\left(\rho_0 \overline{A'v'}\right) + \frac{\partial}{\partial z}\left(\rho_0 \overline{A'w'}\right)$$
$$= \frac{\partial}{\partial t}\left(\rho_0 \overline{A}\right) + \overline{v}\frac{\partial}{\partial y}\left(\rho_0 \overline{A}\right) + \overline{w}\frac{\partial}{\partial z}\left(\rho_0 \overline{A}\right) - \overline{A}\frac{\partial}{\partial z}\left(\rho_0 \overline{w}\right) + \overline{A}\frac{\partial}{\partial z}\left(\rho_0 \overline{w}\right) + \frac{\partial}{\partial y}\left(\rho_0 \overline{A'v'}\right) + \frac{\partial}{\partial z}\left(\rho_0 \overline{A'w'}\right)$$
$$= \frac{\overline{D}}{Dt}\left(\rho_0 \overline{A}\right) + \frac{\partial}{\partial y}\left(\rho_0 \overline{A'v'}\right) + \frac{\partial}{\partial z}\left(\rho_0 \overline{A'w'}\right),$$

with

$$\frac{\overline{D}}{Dt} \equiv \frac{\partial}{\partial t} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}.$$

Next we want to examine the exchange of energy between the eddies and the mean flow. Again, we consider quasi-geostrophic flow in a mid-latitude β-plane. First, determine mean equations in the log-pressure system. We have

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = X$$

We first evaluate

$$\overline{\frac{Du}{Dt}} = \frac{\partial}{\partial t}\left(\overline{u}\right) + \overline{v}\frac{\partial}{\partial y}\left(\overline{u}\right) + \overline{w}\frac{\partial}{\partial z}\left(\overline{u}\right) + \frac{\partial}{\partial y}\left(\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\overline{u'w'}\right),$$

and since we are working in a mid-latitude β-plane, the Coriolis parameter is f_0, $\frac{\partial}{\partial y}\left(\overline{u}\right) = 0$. Also, $\overline{u} \neq \overline{u}(z)$.

$$\therefore \overline{\frac{Du}{Dt}} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial y}\left(\overline{u'v'}\right)$$

For the Coriolis force term $\overline{f_0 v} = f_0\overline{(\overline{v} + v')} = f_0\overline{v}$.

For the turbulent drag force $\overline{X} = \overline{\left(\overline{X} + X'\right)} = \overline{X}$.

For the geopotential term $\overline{\frac{\partial \Phi}{\partial x}} = 0$ since $\frac{\partial \overline{(\,)}}{\partial x} = 0$.

$$\therefore \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - f_0\overline{v} = \overline{X}.$$

The y-component of the momentum equation is

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = Y.$$

Neglecting advection by the mean meridional circulation as well as vertical eddy fluxes in a mid-latitude β-plane

$$\overline{\frac{Dv}{Dt}} = 0 \;\text{ and }\; f_0\overline{(\overline{u} + u')} = f_0\overline{u}$$

The geopotential term $\overline{\frac{\partial \Phi}{\partial y}} = \frac{\partial}{\partial y}\overline{\left(\overline{\Phi} + \Phi'\right)} = \frac{\partial \overline{\Phi}}{\partial y}$.

$Y = 0$ since we only include a turbulent drag force in the x-component of the momentum equation.

$$\therefore f_0 \overline{u} + \frac{\partial \overline{\Phi}}{\partial y} = 0.$$

Next consider the thermodynamic energy equation:

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}\right) \frac{\partial \Phi}{\partial z} + wN^2 = \frac{J\kappa}{H}.$$

Again using the formula for $\rho_0 \overline{\frac{DA}{Dt}}$, and considering the assumptions made above, the mean thermodynamic equation becomes

$$\begin{aligned} \frac{D}{Dt}\left(\frac{\partial \overline{\Phi}}{\partial z}\right) + \overline{w}N^2 &= \frac{\overline{J}\kappa}{H} \\ \therefore \frac{\partial}{\partial t}\left(\frac{\partial \overline{\Phi}}{\partial z}\right) + \frac{\partial}{\partial y}\overline{\left(\frac{\partial \Phi'}{\partial z} v'\right)} + \overline{w}N^2 &= \frac{\overline{J}\kappa}{H} \end{aligned}$$

We have neglected the advection by the mean meridional circulation, $\overline{v}\frac{\partial}{\partial y}\left(\frac{\partial \overline{\Phi}}{\partial z}\right)$, in the above equation. Our mean set of equations in log-pressure coordinates is

$$\begin{aligned} \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - f_0 \overline{v} - \overline{X} &= 0 \\ f_0 \overline{u} + \frac{\partial \overline{\Phi}}{\partial y} &= 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial \overline{\Phi}}{\partial z}\right) + \frac{\partial}{\partial y}\overline{\left(\frac{\partial \Phi'}{\partial z} v'\right)} + \overline{w}N^2 - \frac{\overline{J}\kappa}{H} &= 0. \end{aligned}$$

Next we derive a similar set of dynamical equations for the eddy motion. Again consider

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = X$$

in the mid-latitude β-plane with the Coriolis parameter f_0. Since we are neglecting vertical eddy flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(\overline{u} + u'\right) = \frac{\partial u'}{\partial t} \\ u\frac{\partial u}{\partial x} &= \left(\overline{u} + u'\right)\frac{\partial}{\partial x}\left(\overline{u} + u'\right) \\ &= \overline{u}\frac{\partial}{\partial x}\left(\overline{u} + u'\right) + u'\frac{\partial}{\partial x}\left(\overline{u} + u'\right) \\ &= \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \overline{u}}{\partial x} + u'\frac{\partial u'}{\partial x} \end{aligned}$$

Since $\overline{\frac{\partial \Phi}{\partial x}} = 0$ and since we are developing a linearised set of equations we neglect $u'\frac{\partial u'}{\partial x}$ as a non-linear term.

$$u\frac{\partial u}{\partial x} = \overline{u}\frac{\partial u'}{\partial x}.$$

$$\begin{aligned} v\frac{\partial u}{\partial y} &= \left(\overline{v} + v'\right)\frac{\partial}{\partial y}\left(\overline{u} + u'\right) \\ &= \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \overline{u}}{\partial y} + v'\frac{\partial u'}{\partial y} \end{aligned}$$

We neglect $\overline{v}$ terms in the mid-latitude β-plane at $f = f_0$ as well as the non-linear term $v'\frac{\partial u'}{\partial y}$.

$$\therefore v\frac{\partial u}{\partial y} = v'\frac{\partial \overline{u}}{\partial y}.$$

$$\begin{aligned} \implies \frac{Du}{Dt} &= \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \overline{u}}{\partial y} \\ &= \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)u' + v'\frac{\partial \overline{u}}{\partial y} \end{aligned}$$

$$fv = f_0\left(\overline{v} + v'\right) = f_0 v'$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \overline{\Phi}}{\partial x} + \frac{\partial \Phi'}{\partial x} = \frac{\partial \Phi'}{\partial x}$$

$$X = \overline{X} + X' = X'$$

$$\implies \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)u' + v'\frac{\partial \overline{u}}{\partial y} - f_0 v' + \frac{\partial \Phi'}{\partial x} = X'$$

$$\therefore \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)u' - \left(f_0 - \frac{\partial \overline{u}}{\partial y}\right)v' + \frac{\partial \Phi'}{\partial x} = X'$$

The second momentum equation:

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = Y$$

$$\begin{aligned} \frac{Dv}{Dt} &= \frac{\partial}{\partial t}\left(\overline{v} + v'\right) + \overline{u}\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial v'}{\partial x}\right) + u'\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial v'}{\partial x}\right) + \overline{v}\left(\frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y}\right) + v'\left(\frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y}\right) \\ &= \frac{\partial v'}{\partial t} + \overline{u}\frac{\partial v'}{\partial x} + u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} \end{aligned}$$

Again neglecting non-linear terms:

$$\frac{Dv}{Dt} = \frac{\partial v'}{\partial t} + \overline{u}\frac{\partial v'}{\partial x}$$

$fu = f_0 u'$, $\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi'}{\partial y}$, and $Y = Y'$ in eddy equation.

$$\implies \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)v' + f_0 u' + \frac{\partial \Phi'}{\partial y} = Y'$$

The thermodynamic energy equation

$$\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}\right)\frac{\partial\Phi}{\partial z}+wN^2=\frac{J\kappa}{H}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial z}\right)&=\frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial z}\right)\\
\vec{V}\cdot\vec{\nabla}\frac{\partial\Phi}{\partial z}&=u\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial z}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial z}\right)\\
&=\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial\overline{\Phi}}{\partial z}+\frac{\partial\Phi'}{\partial z}\right)+u'\frac{\partial}{\partial x}\left(\frac{\partial\overline{\Phi}}{\partial z}+\frac{\partial\Phi'}{\partial z}\right)+\overline{v}\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}+\frac{\partial\Phi'}{\partial z}\right)+v'\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}+\frac{\partial\Phi'}{\partial z}\right)\\
&=\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)+u'\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)+v'\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)+v'\frac{\partial}{\partial y}\left(\frac{\partial\Phi'}{\partial z}\right)
\end{aligned}$$

Again, neglecting non-linear terms:

$$\vec{V}\cdot\vec{\nabla}\frac{\partial\Phi}{\partial z}=\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)+v'\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)$$

$wN^2=w'N^2$ and $\dfrac{J\kappa}{H}=\dfrac{J'\kappa}{H}$ in eddy equation

$$\Longrightarrow\ \frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial z}\right)+\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)+v'\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)+w'N^2=\frac{J'\kappa}{H}$$

$$\therefore\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\frac{\partial\overline{\Phi}}{\partial z}+v'\frac{\partial}{\partial y}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)+N^2w'=\frac{\kappa}{H}J'$$

As an eddy equation, the continuity equation becomes

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0w')=0$$

Now that we have these two sets of equations, we examine the exchange of energy between the eddies and the mean flow globally, by first defining a global average

$$\langle\ \rangle\equiv\frac{1}{A}\int_0^\infty\int_0^D\int_0^L(\ )dxdydz,$$

with L the distance around a latitude circle and D the meridional extend of the mid-latitude β-plane. A represents the horizontal area of the β-plane, and so the global average has a length scale $\left(m^3/m^2=m\right)$.

For any quantity Ψ, the atmosphere is confined to a zonal channel in the mid-latitude β-plane with rigid walls.

Note: X' and Y' are zonally varying components of drag owing to turbulence.

At $y=\pm D$,

$$\left\langle\frac{\partial\Psi}{\partial y}\right\rangle=0$$

if Ψ vanishes there.

Also, Ψ remains constant in the zonal direction at $y = \pm D$, and so

$$\left\langle \frac{\partial \Psi}{\partial x} \right\rangle = 0.$$

If Ψ vanishes at the bottom $(z = 0)$ and at the top of the atmosphere $(z \to \infty)$,

$$\left\langle \frac{\partial \Psi}{\partial z} \right\rangle = 0.$$

Revisiting the mean set of momentum equations in log-pressure coordinates, multiplying them respectively with $\rho_0 \overline{u}$ and $\rho_0 \overline{v}$, and adding the results:

$$\left(\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v} + \frac{\partial}{\partial y}(\overline{u'v'}) - \overline{X} \right) \rho_0 \overline{u} = 0$$

$$\left(f_0 \overline{u} + \frac{\partial \overline{\Phi}}{\partial y} \right) \rho_0 \overline{v} = 0$$

$$\rho_0 \overline{u} \frac{\partial \overline{u}}{\partial t} - f_0 \rho_0 \overline{u}\, \overline{v} + \rho_0 \overline{u} \frac{\partial}{\partial y}(\overline{u'v'}) - \rho_0 \overline{u}\overline{X} + f_0 \rho_0 \overline{u}\, \overline{v} + \rho_0 \overline{v} \frac{\partial \overline{\Phi}}{\partial y} = 0$$

$$\therefore \frac{1}{2} \rho_0 \frac{\partial}{\partial t} \overline{u}^2 = -\rho_0 \overline{u} \frac{\partial}{\partial y}(\overline{u'v'}) + \rho_0 \overline{u}\overline{X} - \rho_0 \overline{v} \frac{\partial \overline{\Phi}}{\partial y}$$

Consider

$$\frac{\partial}{\partial y}(\rho_0 \overline{u}\overline{u'v'}) = \rho_0 \overline{u} \frac{\partial}{\partial y}(\overline{u'v'}) + \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y}$$

$$\therefore -\rho_0 \overline{u} \frac{\partial}{\partial y}(\overline{u'v'}) = \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} - \frac{\partial}{\partial y}(\rho_0 \overline{u}\overline{u'v'})$$

and

$$\frac{\partial}{\partial y}\left(\rho_0 \overline{v}\overline{\Phi}\right) = \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} + \rho_0 \overline{v} \frac{\partial \overline{\Phi}}{\partial y}$$

$$\therefore -\rho_0 \overline{v} \frac{\partial \overline{\Phi}}{\partial y} = \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} - \frac{\partial}{\partial y}\left(\rho_0 \overline{v}\overline{\Phi}\right)$$

$$\therefore \frac{1}{2} \rho_0 \frac{\partial}{\partial t} \overline{u}^2 = \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} - \frac{\partial}{\partial y}(\rho_0 \overline{u}\overline{u'v'}) + \rho_0 \overline{u}\overline{X} + \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} - \frac{\partial}{\partial y}\left(\rho_0 \overline{v}\overline{\Phi}\right)$$

Assuming that $\overline{v} = 0$ and $\overline{u'v'} = 0$ for $y \pm D$, after integrating the above equation we get the mean flow kinetic energy

$$\frac{d}{dx} \left\langle \frac{\rho_0 \overline{u}^2}{2} \right\rangle = \left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle + \langle \rho_0 \overline{u}\overline{X} \rangle + \left\langle \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} \right\rangle$$

Consider the continuity equation:

$$\frac{\partial \overline{v}}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial z}(\rho_0 \overline{w}) = 0,$$

since $\frac{\partial\overline{(\ )}}{\partial x} = 0.$

$$\therefore \left\langle \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} \right\rangle = \left\langle \rho_0 \overline{\Phi} \left(-\frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 \overline{w}) \right) \right\rangle$$
$$= - \left\langle \overline{\Phi} \frac{\partial}{\partial z} (\rho_0 \overline{w}) \right\rangle$$

Since

$$\frac{\partial}{\partial z} \left(\overline{\Phi} (\rho_0 \overline{w}) \right) = \overline{\Phi} \frac{\partial}{\partial z} (\rho_0 \overline{w}) + \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z}$$
$$\therefore \left\langle \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} \right\rangle = \left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} - \frac{\partial}{\partial z} \left(\overline{\Phi} (\rho_0 \overline{w}) \right) \right\rangle$$

Assuming that $\rho_0 \overline{w} = 0$ at $z = 0$, and $z \to \infty$, the result is

$$\left\langle \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} \right\rangle = \left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle$$

Since the hydrostatic equation states that

$$\frac{\partial \overline{\Phi}}{\partial z} = \frac{R\overline{T}}{H}$$
$$\therefore \left\langle \rho_0 \overline{\Phi} \frac{\partial \overline{v}}{\partial y} \right\rangle = \frac{R}{H} \left\langle \rho_0 \overline{w} \overline{T} \right\rangle$$

In order to better understand the two terms above, we investigate their SI units. The units of the term on the left hand side is

$$\left(\mathrm{kg\,m^{-3}}\right) \left(\mathrm{m^2\,s^{-2}}\right) \left(\frac{\mathrm{m\,s^{-1}}}{\mathrm{m}}\right)$$

The units of Newton's second law of motion are

$$F = \mathrm{kg\,m\,s^{-2}},$$

where F represents a force.

Therefore the units of the left hand side become $F\,\mathrm{m^{-2}\,s^{-1}}$.

The term on the left hand side therefore represents a force per unit area per unit time. Moreover, take note that $\overline{\Phi}$ at height z is the work required to raise a unit mass to height z and since $\Phi \sim \delta p$, the left hand side term represents work done in terms of vertical pressure changes.

The units of the term on the right hand side are the same as those of the term on the left hand side:

$$\frac{J\,\mathrm{K^{-1}\,kg^{-1}}}{\mathrm{m}} \mathrm{kg\,m^{-3}\,m\,s^{-1}\,K}$$
$$= \mathrm{N\,m\,m^{-3}\,s^{-1}}$$
$$= \mathrm{kg\,m\,s^{-2}\,m^{-2}\,s^{-1}}$$
$$= F\,\mathrm{m^{-2}\,s^{-1}}$$

This right hand side term represents the interrelationship between the zonal-mean vertical mass flux $\rho_0\overline{w}$ (units are $\mathrm{kg\,m^{-2}\,s^{-1}}$), and the zonal-mean temperature, $\overline{T}$. For upward motion $\rho_0\overline{w} > 0$, and for warm air rising $\rho_0\overline{w}\overline{T} > 0$, since $\overline{T}$ (thickness) > 0. For downward motion $\rho_0\overline{w} < 0$, and for sinking cold air $\overline{T} < 0$, therefore $\rho_0\overline{w}\overline{T} > 0$. The right hand side term is thus positive if on average warm air is rising and cold air is sinking so that there is a conversion from potential to kinetic energy.

Recalling that the perturbation available potential energy in the two layer model has the following form

$$\begin{aligned}\frac{\lambda^2}{2}\overline{(\psi_1' - \psi_3')^2} &= \frac{f_0^2}{2\sigma(\delta p)^2}\overline{\left(\frac{\Phi_1'}{f_0} - \frac{\Phi_3'}{f_0}\right)^2} \\ &= \frac{f_0}{2\sigma(\delta p)^2}\overline{\left(\frac{\partial \Phi'}{\partial z}\right)^2}\end{aligned}$$

Here we similarly define the zonal-mean available potential energy as proportional to squared thickness changes divided by the static stability

$$\overline{P} \equiv \frac{1}{2}\left\langle \frac{\rho_0}{N^2}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)^2\right\rangle$$

In order to obtain an expression for the rate of change of zonal-mean available potential energy, multiply the mean thermodynamic energy equation by $\rho_0\left(\frac{\partial\overline{\Phi}}{\partial z}\right)/N^2$

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial\overline{\Phi}}{\partial z}\right) + \frac{\partial}{\partial y}\left(\overline{v'\frac{\partial\Phi'}{\partial z}}\right) + wN^2 - \frac{\overline{J}\kappa}{H}\right]\frac{\rho_0}{N^2}\frac{\partial\overline{\Phi}}{\partial z} = 0$$

$$\therefore \frac{\rho_0}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial\overline{\Phi}}{\partial z}\right) + \frac{\rho_0}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial y}\left(\overline{v'\frac{\partial\Phi'}{\partial z}}\right) + \rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z} - \frac{\rho_0}{N^2}\frac{\overline{J}\kappa}{H}\frac{\partial\overline{\Phi}}{\partial z} = 0$$

Take note that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial\overline{\Phi}}{\partial z}\right)^2\right] &= \frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial\overline{\Phi}}{\partial z}\right)\left(\frac{\partial\overline{\Phi}}{\partial z}\right)\right] \\ &= \frac{1}{2}\left[\frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial\overline{\Phi}}{\partial z}\right) + \frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)\right] \\ &= \frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)\end{aligned}$$

$$\implies \frac{\rho_0}{2N^2}\frac{\partial}{\partial t}\left[\left(\frac{\partial\overline{\Phi}}{\partial z}\right)^2\right] + \frac{\rho_0}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial y}\left(\overline{v'\frac{\partial\Phi'}{\partial z}}\right) + \rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z} - \frac{\rho_0}{N^2}\frac{\overline{J}\kappa}{H}\frac{\partial\overline{\Phi}}{\partial z} = 0$$

After averaging over space, the zonal-mean available potential energy rate is

$$\frac{d}{dt}\left\langle\frac{\rho_0}{2N^2}\left(\frac{\partial\overline{\Phi}}{\partial z}\right)^2\right\rangle = -\left\langle\frac{\rho_0}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\frac{\partial}{\partial y}\left(\overline{v'\frac{\partial\Phi'}{\partial z}}\right)\right\rangle - \left\langle\rho_0\overline{w}\frac{\partial\overline{\Phi}}{\partial z}\right\rangle + \left\langle\frac{\rho_0}{N^2}\frac{\overline{J}\kappa}{H}\frac{\partial\overline{\Phi}}{\partial z}\right\rangle$$

Recall the mean flow kinetic energy evolution

$$\frac{d}{dt}\left\langle \frac{\rho_0 \overline{u}^2}{2} \right\rangle = \left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle + \left\langle \rho_0 \overline{u} \overline{X} \right\rangle + \left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle$$

where the third term on the right is equal and opposite to the second term on the right of the zonal-mean available potential energy evolution equation. Therefore, this term represents a conversion between zonal mean kinetic and potential energies.

The term $\left\langle \frac{\rho_0}{N^2} \frac{\overline{J}\kappa}{H} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle$ represents the interrelationship between temperature (thickness, $\partial\overline{\Phi}/\partial z$) and diabatic heating ($\overline{J}$), and describes the generation of zonal-mean potential energy by diabatic processes. The term $\left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y} \left(\overline{v' \frac{\partial \Phi'}{\partial z}} \right) \right\rangle$ involves meridional ($\partial/\partial y$) and eddy heat flux ($\overline{v'\partial\Phi'/\partial z}$ represents advection or flux) and describes the conversion between zonal-mean and eddy potential energy. The term $\left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle$ of the mean flow kinetic energy to zonal-mean kinetic energy.

Recall that the eddy momentum equations are

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u' - \left(f_0 - \frac{\partial \overline{u}}{\partial y}\right) v' + \frac{\partial \Phi'}{\partial x} = X',$$

and

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) v' + f_0 u' + \frac{\partial \Phi'}{\partial y} = Y';$$

multiply these equations respectively by $\rho_0 u'$ and $\rho_0 v'$ and add the resulting equations.

$$\rho_0 \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) u'^2 - \rho_0 \left(f_0 - \frac{\partial \overline{u}}{\partial y}\right) u'v' + \rho_0 u' \frac{\partial \Phi'}{\partial x} = \rho_0 u' X',$$

and

$$\rho_0 \left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right) v'^2 + \rho_0 f_0 u'v' + \rho_0 v' \frac{\partial \Phi'}{\partial y} = \rho_0 v' Y'$$

Adding these two equations together results in

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_0 u'^2\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(\rho_0 v'^2\right) + \rho_0 \overline{u}\frac{\partial u'^2}{\partial x} + \rho_0 \overline{u}\frac{\partial v'^2}{\partial x} + \rho_0 u'v' \frac{\partial \overline{u}}{\partial y} + \rho_0 u' \frac{\partial \Phi'}{\partial x} + \rho_0 v' \frac{\partial \Phi'}{\partial y} = \rho_0 u' X' + \rho_0 v' Y'$$

Consider

$$\frac{\partial}{\partial x}\left(u'\Phi'\right) = u'\frac{\partial \Phi'}{\partial x} + \Phi'\frac{\partial u'}{\partial x}$$

$$\therefore u'\frac{\partial \Phi'}{\partial x} = \frac{\partial}{\partial x}\left(u'\Phi'\right) - \Phi'\frac{\partial u'}{\partial x}$$

$$\text{Similarly, } v'\frac{\partial \Phi'}{\partial y} = \frac{\partial}{\partial y}\left(v'\Phi'\right) - \Phi'\frac{\partial v'}{\partial y}$$

After applying zonal and global averaging

$$\frac{d}{dt}\left\langle\frac{\rho_0\overline{u'}^2}{2}\right\rangle+\frac{d}{dt}\left\langle\frac{\rho_0\overline{v'}^2}{2}\right\rangle+\left\langle\rho_0\overline{u}\frac{\partial}{\partial x}\left(\overline{u'}^2\right)\right\rangle+\left\langle\rho_0\overline{u}\frac{\partial}{\partial x}\left(\overline{v'}^2\right)\right\rangle+\left\langle\rho_0\overline{u'v'}\frac{\partial\overline{u}}{\partial y}\right\rangle+\left\langle\rho_0\left(\frac{\partial}{\partial x}\overline{(u'\Phi')}-\overline{\Phi'\frac{\partial u'}{\partial x}}\right)\right\rangle$$
$$+\left\langle\rho_0\left(\frac{\partial}{\partial y}\overline{(v'\Phi')}-\overline{\Phi'\frac{\partial v'}{\partial y}}\right)\right\rangle=\left\langle\rho_0\overline{u'X'}\right\rangle+\left\langle\rho_0\overline{v'Y'}\right\rangle$$

Since $\frac{\partial}{\partial x}\overline{(\;)}=0$ and assuming $v'\Phi'=0$ for $y=\pm D$, we get

$$\frac{d}{dt}\left\langle\rho_0\frac{\overline{u'}^2+\overline{v'}^2}{2}\right\rangle+\left\langle\rho_0\overline{u'v'}\frac{\partial\overline{u}}{\partial y}\right\rangle-\left\langle\rho_0\left(\overline{\Phi'\frac{\partial u'}{\partial x}}+\overline{\Phi'\frac{\partial v'}{\partial y}}\right)\right\rangle=\left\langle\rho_0\left(\overline{u'X'+v'Y'}\right)\right\rangle$$

$$\frac{d}{dt}\left\langle\rho_0\frac{\overline{u'}^2+\overline{v'}^2}{2}\right\rangle=\left\langle\rho_0\overline{\Phi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)}\right\rangle-\left\langle\rho_0\overline{u'v'}\frac{\partial\overline{u}}{\partial y}\right\rangle+\left\langle\rho_0\left(\overline{u'X'}+\overline{v'Y'}\right)\right\rangle,$$

the eddy kinetic energy equation.

Recall the eddy thermodynamic energy equation and multiply it with $\frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}$

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial z}\right)+\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)+v'\frac{\partial^2\overline{\Phi}}{\partial y\partial z}+N^2w'-\frac{\kappa}{H}J'\right]\frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}=0$$

$$\frac{\rho_0}{2N^2}\frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial z}\right)^2+\frac{\rho_0\overline{u}}{2N^2}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right)^2+\frac{\rho_0}{N^2}\frac{\partial^2\overline{\Phi}}{\partial y\partial z}v'\frac{\partial\Phi'}{\partial z}+\rho_0w'\frac{\partial\Phi'}{\partial z}-\frac{\rho_0\kappa}{N^2H}J'\frac{\partial\Phi'}{\partial z}=0$$

After applying zonal and global averaging

$$\frac{d}{dt}\left\langle\frac{\rho_0}{2N^2}\overline{\left(\frac{\partial\Phi'}{\partial z}\right)^2}\right\rangle+\left\langle\frac{\rho_0}{N^2}\frac{\partial^2\overline{\Phi}}{\partial y\partial z}\overline{\left(v'\frac{\partial\Phi'}{\partial z}\right)}\right\rangle+\left\langle\rho_0\overline{w'\frac{\partial\Phi'}{\partial z}}\right\rangle+\left\langle\frac{\rho_0\kappa}{N^2H}\overline{J'\frac{\partial\Phi'}{\partial z}}\right\rangle=0$$

$$\therefore\frac{d}{dt}\left\langle\frac{\rho_0}{2N^2}\overline{\left(\frac{\partial\Phi'}{\partial z}\right)^2}\right\rangle=-\left\langle\rho_0\overline{w'\frac{\partial\Phi'}{\partial z}}\right\rangle-\left\langle\frac{\rho_0}{N^2}\frac{\partial^2\overline{\Phi}}{\partial y\partial z}\overline{\left(v'\frac{\partial\Phi'}{\partial z}\right)}\right\rangle+\left\langle\frac{\rho_0\kappa}{N^2H}\overline{J'\frac{\partial\Phi'}{\partial z}}\right\rangle,$$

the eddy potential energy equation.

The first term on the right of the eddy kinetic energy equation when using the eddy continuity equation

$$\begin{aligned}\left\langle\rho_0\overline{\Phi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)}\right\rangle&=\left\langle\rho_0\overline{\Phi'\left(-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0w'\right)\right)}\right\rangle\\&=-\left\langle\overline{\Phi'\frac{\partial}{\partial z}\left(\rho_0w'\right)}\right\rangle\end{aligned}$$

Consider

$$\frac{\partial}{\partial z}\left(\Phi'\left(\rho_0w'\right)\right)=\Phi'\frac{\partial}{\partial z}\left(\rho_0w'\right)+\left(\rho_0w'\right)\frac{\partial\Phi'}{\partial z}$$
$$\therefore-\Phi'\frac{\partial}{\partial z}\left(\rho_0w'\right)=\rho_0w'\frac{\partial\Phi'}{\partial z}-\frac{\partial}{\partial z}\left(\Phi'\left(\rho_0w'\right)\right)$$

$$\implies \left\langle \rho_0 \overline{\Phi' \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right)} \right\rangle = \left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle,$$

after assuming that $w' = 0$ at $z = 0$ $\left(\text{i.e. } \left\langle \rho_0 \frac{\partial}{\partial z} \overline{w'\Phi'} \right\rangle \right)$.

Since the term $\left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle$ is equal to minus the first term on the right of the eddy available potential energy equation, this term expresses the conversion between eddy kinetic and eddy potential energy.

Consider the term $-\left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y} \overline{\left(v' \frac{\partial \Phi'}{\partial z} \right)} \right\rangle$ of the zonal-mean available potential energy. This term is further analysed by considering

$$\frac{\partial}{\partial y} \left[\frac{\partial \Phi'}{\partial z} \left(v' \frac{\partial \Phi'}{\partial z} \right) \right] = \frac{\partial^2 \overline{\Phi}}{\partial y \partial z} \left(v' \frac{\partial \Phi'}{\partial z} \right) + \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y} \left(v' \frac{\partial \Phi'}{\partial z} \right).$$

Since the term on the left vanishes after applying zonal and global averaging because $\left\langle \frac{\partial \Psi}{\partial y} = 0 \right\rangle$ at $y = \pm D$,

$$-\left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y} \overline{\left(v' \frac{\partial \Phi'}{\partial z} \right)} \right\rangle = +\left\langle \frac{\rho_0}{N^2} \frac{\partial^2 \overline{\Phi}}{\partial y \partial z} \overline{\left(v' \frac{\partial \Phi'}{\partial z} \right)} \right\rangle.$$

Therefore, the second term of the eddy potential energy equation is equal to minus the first term on the right of the zonal-mean potential energy, and therefore is the conversion term between eddy and zonal-mean available potential energy.

Next we summarise the four energy equations we derived above as follows:

Mean kinetic energy $\left(\overline{K} \equiv \left\langle \frac{\rho_0 \overline{u}^2}{2} \right\rangle \right)$

$$\frac{d}{dt} \left\langle \frac{\rho_0 \overline{u}^2}{2} \right\rangle = \left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle + \left\langle \rho_0 \overline{u} \overline{X} \right\rangle + \left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle$$

Mean available potential energy $\left(\overline{P} \equiv \left\langle \frac{\rho_0}{2N^2} \left(\frac{\partial \overline{\Phi}}{\partial z} \right)^2 \right\rangle \right)$

$$\frac{d}{dt} \left\langle \frac{\rho_0}{2N^2} \left(\frac{\partial \overline{\Phi}}{\partial z} \right)^2 \right\rangle = -\left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle + \left\langle \frac{\rho_0}{N^2} \frac{\overline{J} \kappa}{H} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle + \left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y} \overline{\left(v' \frac{\partial \Phi'}{\partial z} \right)} \right\rangle$$

Eddy kinetic energy $\left(K' \equiv \left\langle \rho_0 \frac{\overline{u'}^2 + \overline{v'}^2}{2} \right\rangle \right)$

$$\frac{d}{dt} \left\langle \rho_0 \frac{\overline{u'}^2 + \overline{v'}^2}{2} \right\rangle = \left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle - \left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle + \left\langle \rho_0 \left(\overline{u'X'} + \overline{v'Y'} \right) \right\rangle$$

Eddy available potential energy $\left(P' \equiv \left\langle \frac{\rho_0}{2N^2} \overline{\left(\frac{\partial \Phi'}{\partial z}\right)^2} \right\rangle\right)$

$$\frac{d}{dt}\left\langle \frac{\rho_0}{2N^2} \overline{\left(\frac{\partial \Phi'}{\partial z}\right)^2} \right\rangle = -\left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle - \left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) \right\rangle + \left\langle \frac{\rho_0 \kappa}{N^2 H} \overline{J' \frac{\partial \Phi'}{\partial z}} \right\rangle$$

The term that represent energy transformations are

$$\left[\overline{P} \bullet \overline{K}\right] \equiv \left\langle \rho_0 \overline{w} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle$$

$$\left[P' \bullet K'\right] \equiv \left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle$$

$$\left[K' \bullet \overline{K}\right] \equiv \left\langle \rho_0 \overline{u'v'} \frac{\partial \overline{u}}{\partial y} \right\rangle$$

$$\left[P' \bullet \overline{P}\right] \equiv \left\langle \frac{\rho_0}{N^2} \frac{\partial \overline{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) \right\rangle$$

The terms describing energy sources are

$$\overline{R} \equiv \left\langle \frac{\rho_0}{N^2} \frac{\overline{J}\kappa}{H} \frac{\partial \overline{\Phi}}{\partial z} \right\rangle \quad \text{and} \quad R' \equiv \left\langle \frac{\rho_0 \kappa}{N^2 H} \overline{J' \frac{\partial \Phi'}{\partial z}} \right\rangle$$

The terms describing energy sinks are

$$\overline{\varepsilon} \equiv \left\langle \rho_0 \overline{u}\overline{X} \right\rangle \quad \text{and} \quad \varepsilon' \equiv \left\langle \rho_0 \left(\overline{u'X'} + \overline{v'Y'}\right) \right\rangle$$

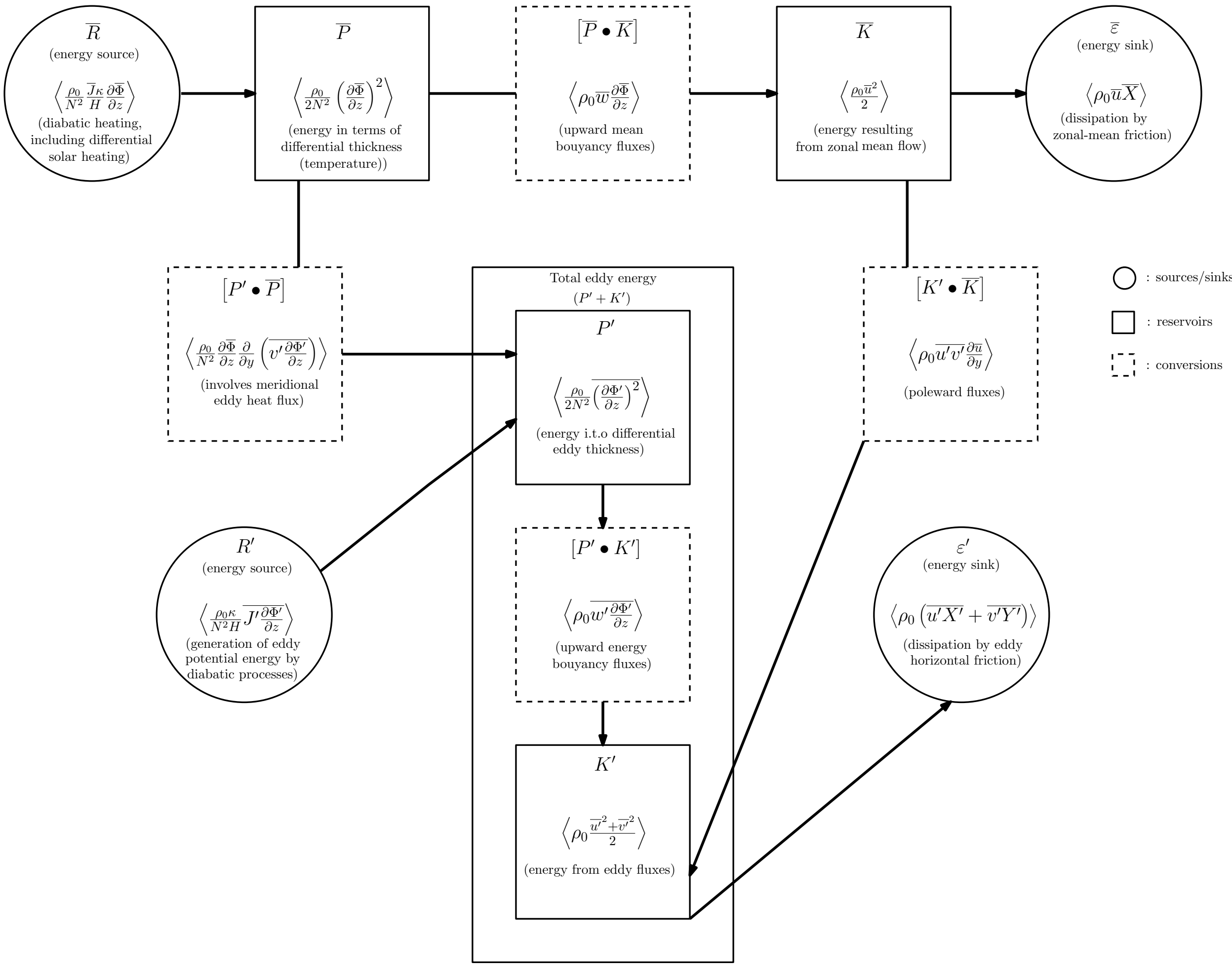
The four energy equations can then be simplified

$$\begin{aligned}
\frac{d}{dt}\overline{K} &= \left[K' \bullet \overline{K}\right] + \overline{\varepsilon} + \left[\overline{P} \bullet \overline{K}\right] \\
\frac{d}{dt}\overline{P} &= -\left[\overline{P} \bullet \overline{K}\right] + \overline{R} + \left[P' \bullet \overline{P}\right] \\
\frac{d}{dt}K' &= \left[P' \bullet K'\right] - \left[K' \bullet \overline{K}\right] + \varepsilon' \\
\frac{d}{dt}P' &= -\left[P' \bullet K'\right] - \left[P' \bullet \overline{P}\right] + R'
\end{aligned}$$

Note on notation: $[A \bullet B]$ means that there is an energy conversion from energy form A to energy form B.

The equation for the rate of change of total energy becomes

$$\begin{aligned}
\frac{d}{dt}\left(\overline{K} + K' + \overline{P} + P'\right) &= \overline{\varepsilon} + \overline{R} + \varepsilon' + R' \\
&= \underbrace{\overline{R} + R'}_{\text{sources}} + \underbrace{\overline{\varepsilon} + \varepsilon'}_{\text{dissipation}}
\end{aligned}$$



For adiabatic processes (take note that synoptic-scale motions are approximately adiabatic outside regions of active precipitation) the diabatic heat rates can be ignored with the result that $\overline{R}$ and R' vanish. Moreover, when there is no dissipation of kinetic into thermal energy as a result of turbulent drag force causing friction, then $\overline{\varepsilon}$ and ε' vanish. The result is that the total energy is conserved, i.e. $\dfrac{d}{dt}\left(\overline{K}+K'+\overline{P}+P'\right)=0.$

In the long-term $\dfrac{d}{dt}\left(\overline{K}+K'+\overline{P}+P'\right)$ must vanish, so that $\overline{R}+R'=-(\overline{\varepsilon}+\varepsilon')$. This equation implies that the production of available potential energy through zonal-mean and eddy diabatic processes, must be able to balance the sum of the mean and eddy kinetic energy dissipation.

Explaining two specific integral simplifications

Referring to Holton, James R.: Introduction to Dynamic Meteorology, 4th edition, 2004, Elsevier.

The equations (8.40), on page 248, call on us to evaluate the integral

$$\int_0^L \cos k(x + x_0 - ct) \sin k(x - ct) dx,$$

and, as a consequence, also the integral

$$\int_0^L [\sin k(x - ct)]^2 \, dx.$$

L is to be taken as the period of the sine and cosine wave expressions. Hence,

$$L = \frac{2\pi}{k}$$

or

$$L = \frac{2\pi n}{k},$$

for some positive integer n.

Preliminaries

Pr. 1

Compound angle identities:

$$\begin{aligned}
\sin(B + C) &= \sin B \cos C + \cos B \sin C \\
\sin(B - C) &= \sin B \cos C - \cos B \sin C \\
\cos(B + C) &= \cos B \cos C - \sin B \sin C \\
\cos(B - C) &= \cos B \cos C + \sin B \sin C
\end{aligned}$$

The following double angle identities are obtained from these, by setting $C = B$:

$$\begin{aligned}\sin 2B &= 2\sin B\cos B\\ \cos 2B &= \cos^2 B - \sin^2 B\\ &= 2\cos^2 B - 1\\ &= 1 - 2\sin^2 B\end{aligned}$$

because $\sin^2 B + \cos^2 B = 1$.

Pr. 2

The analytic sine and cosine functions:

In order to be able to do calculus with wave functions, it was found to be necessary to introduce radian measure. The trigonometric concept 'angle' is modelled (or replaced) by a pure real variable that corresponds to the arc length of a circular arc with radius $= 1$ unit, subtending the angle at its centre.

Since a circle with radius $r = 1$ has circumference $P = 2\pi$, we obtain for the analytic $\mathbb{R}$-$\mathbb{R}$-functions $\sin$ and $\cos$:

$$\begin{aligned}\sin(0) &= 0\\ \sin\left(\frac{\pi}{2}\right) &= 1\\ \sin\pi &= 0\\ \sin\left(\frac{3\pi}{2}\right) &= -1\\ \sin 2\pi &= 0\end{aligned}$$

$$\begin{aligned}\cos(0) &= 1\\ \cos\left(\frac{\pi}{2}\right) &= 0\\ \cos\pi &= -1\\ \cos\left(\frac{3\pi}{2}\right) &= 0\\ \cos 2\pi &= 1\end{aligned}$$

At the annoying cost of having to express the independent variable in terms of π in all applications that involve waves, we obtain the happy results that

$$\begin{aligned}&\frac{d}{dx}\sin x = \cos x\\ &\frac{d}{dx}\sin k(x-v) = k\cos k(x-v)\\ &\frac{d}{dx}\cos x = -\sin x\\ &\frac{d}{dx}\cos k(x-v) = -k\sin k(x-v)\end{aligned}$$

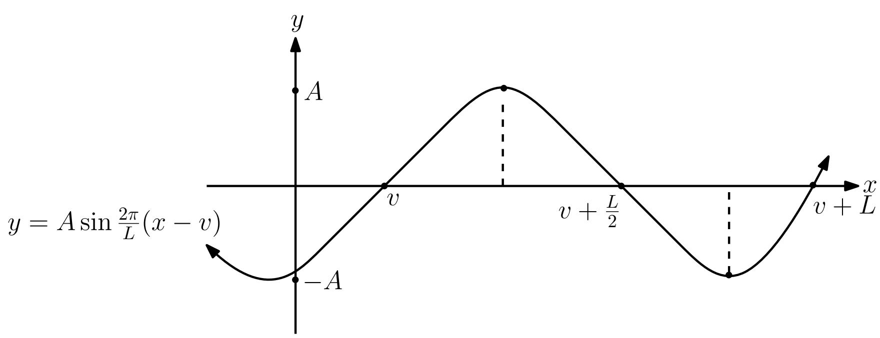
where k and v are independent of x.

Not only do integration and differential equations become humanly consumable, but many more satisfying consequences follow, in particular, we get the power series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Pr. 3

The xy graph of a simple wave:

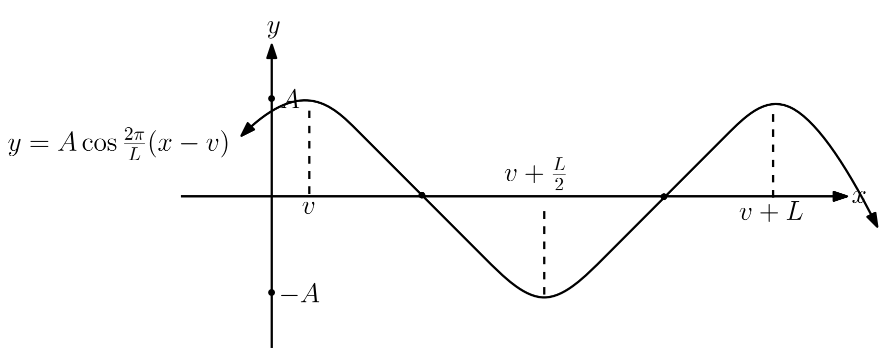


The analytic functions sin and cos have amplitude $= 1$, and period $= 2\pi$. In order to model more general waves by means of these functions, the equations $y = \sin x$, $y = \cos x$ are easily adapted.

It is customary to introduce a parameter, say k, related to the period, L, such that

$$kL = 2\pi,\ k = \frac{2\pi}{L},\ L = \frac{2\pi}{k},$$

The adaptation $y = A\sin\frac{2\pi}{L}(x - v) = A\sin k(x - v)$ is shown in the sketch above, for a sine-based application. For a cosine-based application, see the sketch below:



In general the xy-graph of the equation $y = A\sin k(x - v)$ is a sine wave with period $\dfrac{2\pi}{k}$ and amplitude A, with $y_{\text{max}} = A$, $y_{\text{min}} = -A$, and zeroes $y = 0$ at $x = v$, $v \pm \dfrac{L}{2}$, $v \pm L, \cdots$, where $L =$ period $= \dfrac{2\pi}{k}$. The parameters k, A and L are understood to be constants (independent of x).

The same applies to the cosine equation $y = A\cos k(x - v)$, except that $x = v$, $v \pm L$, $v \pm 2L, \cdots$, now correspond to the points at which $y = A$, while $x = v \pm \frac{L}{2}$, $v \pm \frac{3L}{2}, \cdots$ give the points at which $y = -A$.

So given a simple wave with amplitude A, period L, and offset v, set $k = \frac{2\pi}{2}$; then the xy-graph of the wave is $y = A\sin k(x - v)$ or $y = A\cos k(x - v)$ depending on the role of the parameter v.

Pr. 4

The essence of a periodic function:

When $kL = 2\pi$, so that L is the period, then $\sin k(a + L) = \sin k(a)$, and $\cos k(a + L) = \cos k(a)$, for all a-values.

This can be shown by means of the compound angle identities (Pr. 1), but it is actually obvious from the graphs.

In the work that follows, we will actually have occasion to look at the related facts that $\cos 2k(L - ct) - \cos 2k(0 - ct) = 0$, and $\sin 2k(L - ct) - \sin 2k(0 - ct) = 0$, the period being $\frac{L}{2}$ for these wave functions.

Pr. 5

Two useful derivatives, with corresponding indefinite integrals:

<u>Pr. 5.1</u>

If the parameters $k,\ c,\ t$ are independent of the variable x, then

$$\begin{aligned}\frac{d}{dx}\left[\cos 2k(x - ct)\right] &= -2k\sin 2k(x - ct)\\ &= -4k\sin 2k(x - ct)\cos 2k(x - ct)\end{aligned}$$

by Pr. 1.

Therefore

$$\int -4k\left[\sin 2k(x - ct)\right]\left[\cos 2k(x - ct)\right]dx = \cos 2k(x - ct) + C$$

Furthermore, if $kL = 2\pi$, then it follows that

$$\int_0^L \sin k(x - ct)\cos k(x - ct)dx = 0$$

by Pr. 4.

<u>Pr. 5.2</u>

If the parameters $k,\ c,\ t$ are independent of the variable x, then

$$\begin{aligned}\frac{d}{dx}\left[\sin 2k(x - ct)\right] &= 2k\cos 2k(x - ct)\\ &= 2k\left[1 - 2\sin^2 k(x - ct)\right] \quad \text{by Pr. 1}\\ &= 2k - 4k\sin^2 k(x - ct)\end{aligned}$$

Therefore

$$\int \left[2k - 4k\sin^2 k(x-ct)\right] dx = \sin 2k(x-ct) + C$$

Furthermore, if $kL = 2\pi$, then it follows that

$$\int_0^L 2k - 4k\sin^2 k(x-ct)dx = 0$$

by Pr. 4.

Holton's integrals

By Pr. 1:

$$\begin{aligned}\cos k(x+x_0-ct) &= \cos\left[kx_0 + k(x-ct)\right] \\ &= \cos kx_0 \cos k(x-ct) - \sin kx_0 \sin k(x-ct),\end{aligned}$$

therefore, multiplying both sides with $\sin k(x-ct)$:

$$\cos k(x+x_0-ct)\sin k(x-ct) = \cos kx_0\left[\sin k(x-ct)\cos k(x-ct)\right] - \sin kx_0 \sin^2 k(x-ct)$$

We can now attempt the integral:

$$\begin{aligned}\int_0^L \cos k(x+x_0-ct)\sin k(x-ct)dx &= \cos kx_0 \int_0^L \sin k(x-ct)\cos k(x-ct)dx - \sin kx_0 \int_0^L \sin^2 k(x-ct)dx \\ &= -\sin kx_0 \int_0^L \sin^2 k(x-ct)dx,\end{aligned}$$

because the first term $= 0$ by Pr. 5.1.

This result agrees with (8.40) as given by Holton.

We continue with the second integral:

$$\begin{aligned}\int_0^L \sin^2 k(x-ct)dx &= -\frac{1}{4k}\int_0^L -4k\sin^2 k(x-ct)dx \\ &= -\frac{1}{4k}\int_0^L -2kdx - \frac{1}{4k}\int_0^L \left[2k - 4k\sin^2 k(x-ct)\right] dx \\ &= \frac{2k}{4k}\int_0^L 1dx + 0 \text{ by Pr. 5.2} \\ &= \frac{1}{2}(L-0) \\ &= \frac{L}{2}\end{aligned}$$

Again, we get it right, according to Holton!