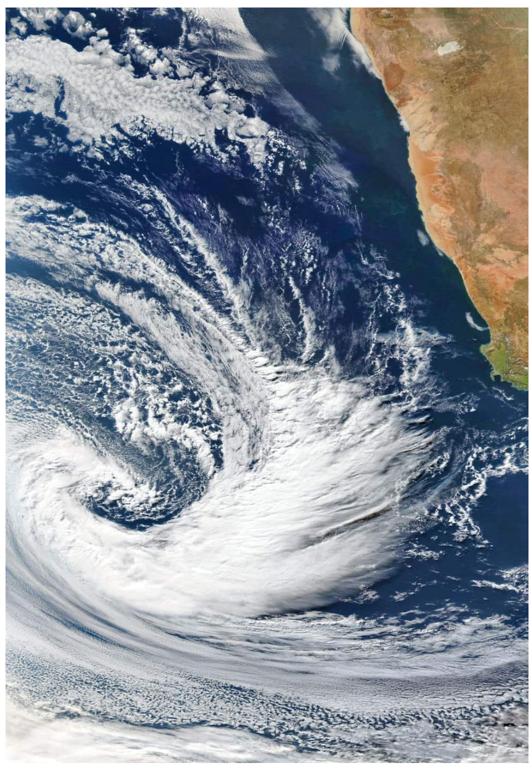
SYNOPTIC-SCALE CIRCULATION AND VORTICITY IN MID-LATITUDES



Willem A. Landman
Donald Nkabinde
Michael Barnes

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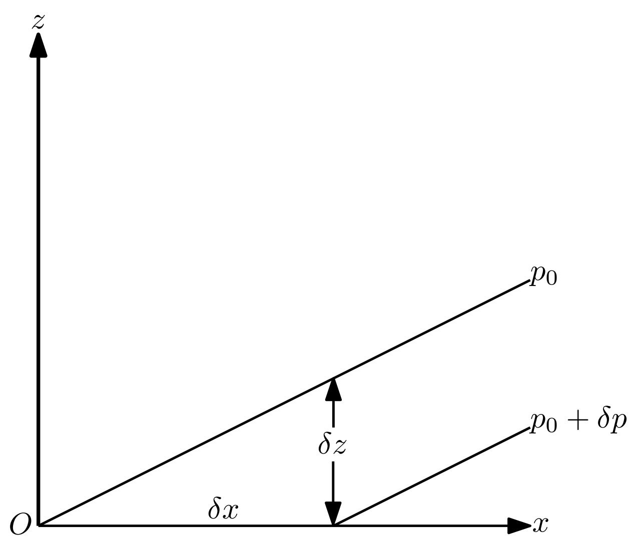
Pressure as a vertical coordinate

Transformation of the horizontal pressure **gradient** force from height to pressure coordinates:



$$\left[\frac{(p_0+\delta p)-p_0}{\delta x}\right]_z = \left[\frac{(p_0+\delta p)-p_0}{\delta z}\right]_x \left(\frac{\delta z}{\delta x}\right)_p$$

Limit as $\delta z \to 0$: $\left[\frac{(p_0+\delta p)-p_0}{\delta z}\right]_x \to -\left(\frac{\partial p}{\partial z}\right)_x$

Limit as $\delta x \to 0$: $\left[\frac{(p_0+\delta p)-p_0}{\delta x}\right]_z \to \left(\frac{\partial p}{\partial x}\right)_z$

$$\therefore \left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

Hydrostatic equation: $\frac{\partial p}{\partial z} = -\rho g$

$$\therefore \left(\frac{\partial p}{\partial x}\right)_z = -(-\rho g)\left(\frac{\partial z}{\partial x}\right)_p$$

$$\therefore -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p$$

Recall the geopotential

$$\Phi = \int_0^z g dz \implies \partial\Phi = g\partial z$$

$$\therefore -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$

Isobaric: Characterised by equal or constant pressure, with respect to either space or time.

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p \tag{1.25}$$

and

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p \tag{1.26}$$

$\implies$ In the isobaric coordinate system, the horizontal pressure gradient force is measured by the gradient of **geopotential** at constant pressure.

Advantage of isobaric system: density is not explicit in the pressure gradient force.

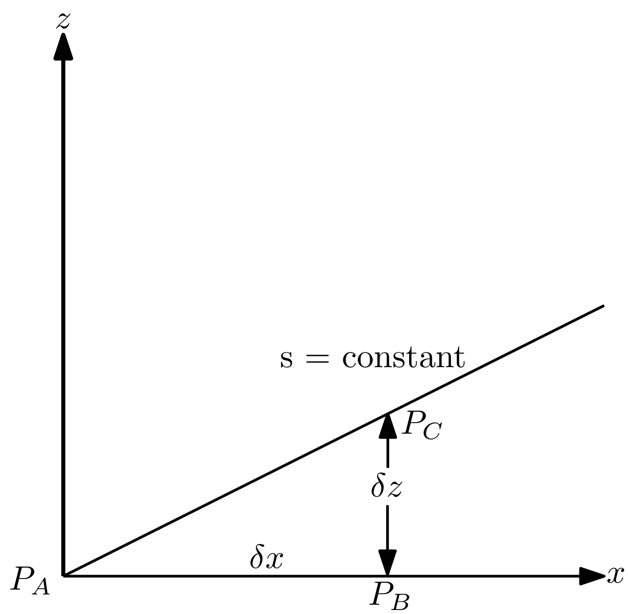
Note:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z - \frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\frac{1}{\rho}\overline{\nabla}_z p$$

$$-\left(\frac{\partial \Phi}{\partial x}\right)_p - \left(\frac{\partial \Phi}{\partial y}\right)_p = -\overline{\nabla}_p \Phi$$

A generalized vertical coordinate

Aim: To obtain a general expression for the horizontal pressure gradient; which is applicable to any vertical coordinate $s = s(x, y, z, t)$



Gradient: $\dfrac{P_C - P_B}{\delta x}$

$$\frac{P_C - P_B}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x}$$

$$\frac{P_C - P_B}{\delta x} - \frac{P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_B - P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_A}{\delta x} - \frac{P_B}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} - \frac{P_A}{\delta x}$$

$$\frac{P_C - P_A}{\delta x} = \frac{P_C - P_B}{\delta z}\frac{\delta z}{\delta x} + \frac{P_B - P_A}{\delta x}$$

$$\begin{aligned} P_C - P_A &: \text{along diagonal where } s \text{ is constant} \\ \frac{\delta z}{\delta x} &: \text{its diagonal is constant } s \\ P_B - P_A &: \text{along } x \text{ where } z \text{ is constant} \end{aligned}$$

Taking the limits as δx, $\delta z \to 0$

$$\implies \left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \tag{1.27}$$

Identity: $\dfrac{\partial p}{\partial z} = \left(\dfrac{\partial s}{\partial z}\right)\left(\dfrac{\partial p}{\partial s}\right)$

$$\begin{aligned} \therefore \left(\frac{\partial p}{\partial x}\right)_s &= \left(\frac{\partial s}{\partial z}\right)\left(\frac{\partial p}{\partial s}\right)\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \\ &= \left(\frac{\partial p}{\partial x}\right)_z + \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s\left(\frac{\partial p}{\partial s}\right) \end{aligned} \tag{1.28}$$

Basic equations in isobaric coordinates

The horizontal momentum equation

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.24}$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.25}$$

In vertical form:

$$\frac{D\overline{V}}{Dt} + f\overline{k} \times \overline{V} = -\frac{1}{\rho}\overline{\nabla} p \tag{3.1}$$

where $\overline{V} = \bar{i}u + \bar{j}v$ is the horizontal velocity.

From page 2:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \Phi}{\partial x}\right)_p \text{ and } -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p$$

$$\frac{D\overline{V}}{Dt} + f\overline{k} \times \overline{V} = -\overline{\nabla}_p \Phi \qquad (\delta p < 0)$$

$\overline{\nabla}_p$: **horizontal** gradient operator (p held constant)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p}$$
$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

where $\omega = \frac{Dp}{Dt}$ is the "omega" vertical motion, the pressure change following the motion.

Consider equation (2.24) again:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

with fv the Coriolis force and $\frac{1}{\rho}\frac{\partial p}{\partial x}$ the pressure gradient force (Pgf).

On the synoptic scale, the order of magnitude of the meridional wind speed, v, is about $10\,\mathrm{m\,s^{-1}}$. Therefore, the order of magnitude of the Coriolis force, $|fv|$, is $\left(10^{-4}\,\mathrm{s^{-1}}\right)\left(10\,\mathrm{m\,s^{-1}}\right)$, which is $10^{-3}\,\mathrm{m\,s^{-2}}$.

Also on the synoptic scale, the Pgf, $\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ has a scale of approximately

$$\begin{aligned}\frac{1}{1\,\mathrm{kg\,m^{-3}}}\frac{10\,\mathrm{hPa}}{1000\,\mathrm{km}} &= \frac{\left(\mathrm{kg^{-1}\,m^{3}}\right)\left(1000\,\mathrm{N\,m^{-2}}\right)}{1\;000\;000\,\mathrm{m}}\\ &= 10^{-3}\,\mathrm{kg^{-1}\,kg\,m\,s^{-2}}\\ &= 10^{-3}\,\mathrm{m\,s^{-2}}\end{aligned}$$

The Coriolis force and Pgf are, therefore, of similar scales. We can thus infer that $\dfrac{Du}{Dt}$ (acceleration) must be small, provided that significant flow curvature is absent, in order for the two forces to be in balance.

Geostrophic wind

$\overline{V}_g = \bar{i}u_g + \bar{j}v_g$

Vectorial form of the geostrophic wind:

$$\overline{V}_g \equiv \overline{k} \times \frac{1}{\rho f}\overline{\nabla} p$$

$$\begin{aligned}\therefore f\overline{V}_g &= \overline{k} \times \frac{1}{\rho}\overline{\nabla} p \qquad \left(-\frac{1}{\rho}\delta p = g\delta z = \delta\Phi\right)\\ &= \underbrace{\overline{k} \times \overline{\nabla}_p \Phi}_{\text{No density term!}}\end{aligned} \tag{3.4}$$

Thus, a given **geopotential gradient** implies the same geostrophic wind at any height, whereas a given **horizontal pressure gradient** implies different geostrophic wind values depending on the density.

From the vector form of the geostrophic wind and the definition of ageostrophic wind:

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \quad \text{and} \quad v_a = v - v_g$$

$$\begin{aligned}\therefore v_a &= v - \frac{1}{\rho f}\frac{\partial p}{\partial x}\\ \therefore fv_a &= fv - \frac{1}{\rho}\frac{\partial p}{\partial x}\end{aligned}$$

Ageostrophic flow must therefore also be small on the synoptic scale.

As a result of the approximate balance between the Coriolis force and Pgf, both acceleration and ageostrophic motion are small on the synoptic scale.

Additionally, it can be shown that for constant Coriolis parameter ($f = f_0$) that the geostrophic wind ($\overline{V}_g$) is non divergent. This implies that the flow is purely horizontal (i.e., $\omega = 0$).

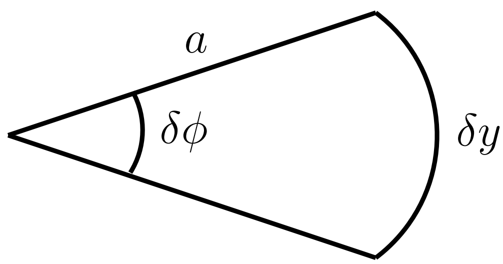
Exercise: The definition of the geostrophic wind in vector form is

$$f\overline{V}_g = \overline{k} \times \overline{\nabla}_p \Phi$$

Derive the divergence of the geostrophic wind for BOTH a constant and variable definition of the Coriolis parameter. For a variable Coriolis parameter, first show that

$$\overline{\nabla} \cdot \overline{V}_g = -\frac{\beta}{f} v_g,$$

then consider



where $a\ (= R_E)$ is the radius of the Earth, ϕ is the latitude and y the length along a latitude circle, to show that for a variable Coriolis parameter the divergence of geostrophic wind is equal to

$$-v_g \frac{\cot \phi}{R_E}$$

Solution:

For constant Coriolis parameter:

$$\begin{aligned}
f\overline{V}_g &= \overline{k} \times \overline{\nabla}\Phi \\
\overline{\nabla} \cdot (f\overline{V}_g) &= \overline{\nabla} \cdot \left(\overline{k} \times \left(\frac{\partial}{\partial x}\overline{i} + \frac{\partial}{\partial y}\overline{j}\right) \Phi\right) \\
f\overline{\nabla} \cdot \overline{V}_g &= \overline{\nabla} \cdot \left(\frac{\partial}{\partial x}\overline{j} - \frac{\partial}{\partial y}\overline{i}\right) \Phi \\
&= \left(\frac{\partial}{\partial x}\overline{i} + \frac{\partial}{\partial y}\overline{j}\right) \cdot \left(-\frac{\partial}{\partial y}\overline{i} + \frac{\partial}{\partial x}\overline{j}\right) \Phi \\
&= \left(-\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y \partial x}\right) \Phi \\
\therefore \overline{\nabla} \cdot \overline{V}_g &= 0
\end{aligned}$$

For variable Coriolis parameter:

We have shown above that $\overline{\nabla} \cdot \left[\overline{k} \times \overline{\nabla}\Phi\right] = 0$

$$\begin{aligned}
\overline{\nabla} \cdot (f\overline{V}_g) &= 0 \\
\therefore \overline{\nabla} \cdot (fu_g\overline{i} + fv_g\overline{j}) &= 0 \\
\therefore \left(\frac{\partial}{\partial x}\overline{i} + \frac{\partial}{\partial y}\overline{j}\right) \cdot (fu_g\overline{i} + fv_g\overline{j}) &= 0 \\
\therefore \frac{\partial}{\partial x}(fu_g) + \frac{\partial}{\partial y}(fv_g) &= 0 \\
\therefore \frac{\partial f}{\partial x}u_g + f\frac{\partial u_g}{\partial x} + \frac{\partial f}{\partial y}v_g + f\frac{\partial v_g}{\partial y} &= 0 \\
\therefore f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) &= -\frac{\partial f}{\partial y}v_g \\
\therefore f\overline{\nabla} \cdot \overline{V}_g &= -\beta v_g \\
\therefore \overline{\nabla} \cdot \overline{V}_g &= -\frac{\beta}{f}v_g \qquad \text{(A)}
\end{aligned}$$

From the figure

$$\begin{aligned}
\delta y &= a\delta\phi \\
\therefore \frac{1}{\delta y} &= \frac{1}{a}\frac{1}{\delta\phi}
\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{\partial f}{\partial y} \\
&= \frac{1}{a}\frac{\partial f}{\partial \phi} \\
&= \frac{1}{a}\frac{\partial}{\partial \phi}(2\Omega\sin\phi) \\
&= \frac{2\Omega}{a}\cos\phi
\end{aligned}$$

$$\begin{aligned}
\therefore \overline{\nabla} \cdot \overline{V}_g &= -\frac{2\Omega}{a}\cos\phi(2\Omega\sin\phi)^{-1}v_g \\
&= -\frac{v_g}{a}\frac{\cos\phi}{\sin\phi} \\
&= -v_g\frac{\cot\phi}{a} \\
&= -v_g\frac{\cot\phi}{R_E} \qquad \text{(B)}
\end{aligned}$$

Equation (A) can be analysed in order show the variability of divergence with latitude. The divergence of $\overline{V}_g$ is directly proportional to β and indirectly proportional to f. Figure 1 shows the variability of the Coriolis parameter, f, with latitude. Many features of f become immediately apparent. Firstly, as we know, the

Coriolis parameter is negative in the SH and positive in the NH. The magnitude of the Coriolis parameter reaches a maximum at the poles and is zero at the equator. Values of the Coriolis parameter are of the order $10^{-4}\,\mathrm{s}^{-1}$. From Equation (A), it can be seen that the equation is undefined at the equator where $f = 0$. However as $|f| \to 0$ close to the equator, it follows from equation (A) that the divergence becomes large and $\left|\overline{\nabla} \cdot \overline{V}_g\right| \to \infty$. Conversely, as $|f|$ increases towards its maximum value, the divergence of $\overline{V}_g$ becomes smaller such that $\left|\overline{\nabla} \cdot \overline{V}_g\right| \to 0$. For completeness, the β-term is also shown in Figure 2. The magnitude of the β-term increases to a maximum towards the equator and decreases towards the poles. Note that β is very small and of the order of $10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$. The contribution of β to the amount of divergence is therefore similar to that of f where it contributes to a greater magnitude of divergence towards the equator whilst a lesser magnitude of divergence close to the poles.

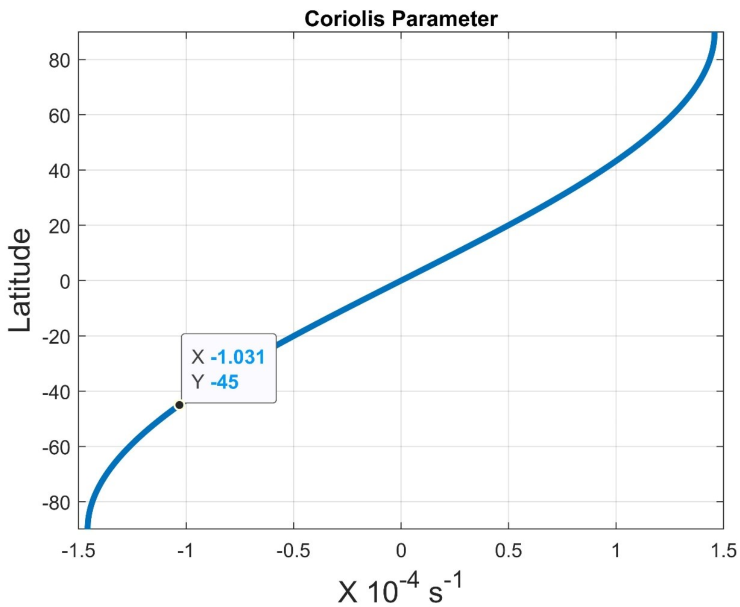


Figure 1: The variability of the Coriolis parameter with latitude.

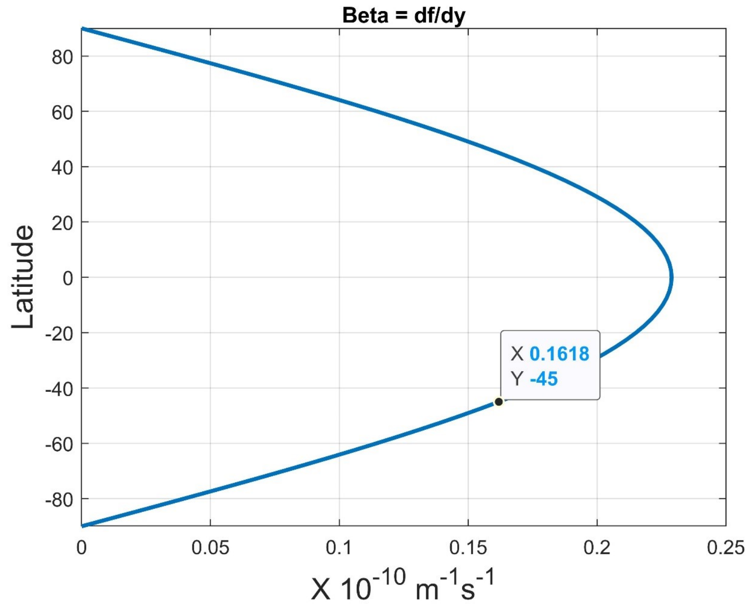


Figure 2: The variation of beta with latitude. Take note that the value of β is of the order $10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$ at $45°$S.

A calculation of the divergence term in Equation (B) provides corroboration of our analysis of Equation (A). The variability of divergence as given by Equation (A) is plotted in Figure 3 below for a number of different v_g values.

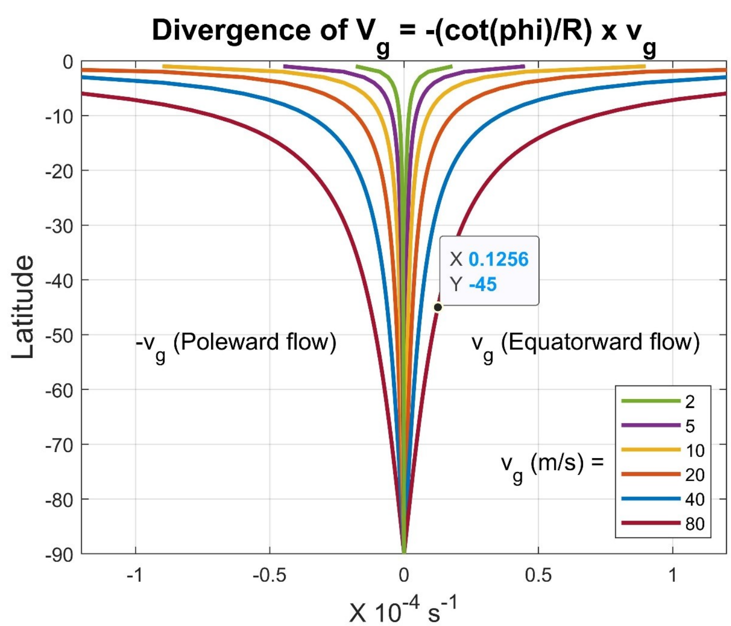


Figure 3: The divergence of the geostrophic wind for a Coriolis parameter that is not constant.

As shown mathematically above, Figure 3 shows that for any v_g the divergence of $\overline{V}_g \to \infty$ as the latitude $\to 0$. Conversely, the divergence of $\overline{V}_g \to 0$ as latitude $\to -90°$. For equatorward flow ($v_g > 0$), divergence occurs, whilst for poleward flow ($v_g < 0$), convergence (negative divergence) occurs.

Importantly, the amount of divergence is small for the majority of the hemisphere, with the exception of flow near the equator. Near the equator, the Coriolis force is very weak or close to zero and thus the geostrophic flow is divergent (i.e., the geostrophic approximation does not hold here since the Coriolis and pressure gradient forces are not in balance). The lack of Coriolis force at the equator is the primary reason why Tropical Cyclones cannot form close to the equator.

The continuity equation

Lagrangian control volume: $\delta V = \delta x \delta y \delta z$

Hydrostatic equation:

$$\begin{aligned} \frac{\delta p}{\delta z} &= -\rho g \\ \therefore \delta z &= -\frac{1}{\rho g}\delta p \\ \therefore \delta V &= -\frac{1}{\rho g}\delta x \delta y \delta p \end{aligned}$$

Mass, conserved following the motion:

$$\begin{aligned} \delta M &= \rho \delta V \\ \therefore \delta M &= -\frac{1}{g}\delta x \delta y \delta p \end{aligned}$$

Thus,

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

The last expression follows from the conservation of mass where $\frac{D}{Dt}(\delta M) = 0 \implies \frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{0}{\delta M}$ Therefore,

$$\begin{aligned} \frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) &= 0 \\ \frac{g}{\delta x \delta y \delta p}\frac{\delta x \delta y}{g}\frac{D}{Dt}\delta p + \frac{g}{\delta x \delta y \delta p}\frac{\delta x \delta p}{g}\frac{D}{Dt}\delta y + \frac{g}{\delta x \delta y \delta p}\frac{\delta y \delta p}{g}\frac{D}{Dt}\delta x &= 0 \qquad \text{(Chain rule)} \\ \frac{1}{\delta x}\frac{D}{Dt}\delta x + \frac{1}{\delta y}\frac{D}{Dt}\delta y + \frac{1}{\delta p}\frac{D}{Dt}\delta p &= 0 \\ \frac{1}{\delta x}\delta \underbrace{\left(\frac{Dx}{Dt}\right)}_{=u} + \frac{1}{\delta y}\delta \underbrace{\left(\frac{Dy}{Dt}\right)}_{=v} + \frac{1}{\delta p}\delta \underbrace{\left(\frac{Dp}{Dt}\right)}_{=\omega} &= 0 \\ \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} &= 0 \end{aligned}$$

Taking limits as $\delta x, \delta y, \delta p \to 0$

And $\delta x, \delta y$ are evaluated at constant pressure

$$\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p}}_{\text{No density, no time derivative!}} = 0$$

$$\frac{\partial \omega}{\partial p} = -\overline{\nabla} \cdot \overline{V}$$

Horizontal divergence: $\overline{\nabla} \cdot \overline{V} > 0 \implies \dfrac{\partial \omega}{\partial p} < 0$, vertical squashing.

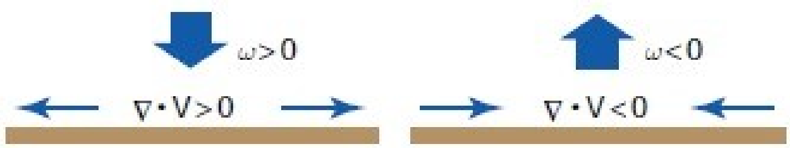


Figure 4: Algebraic signs of ω in the midtroposphere associated with convergence and divergence in the lower troposphere. [Source: Wallace, J.M. and Hobbs, P.V. (2006). Atmospheric Science: An Introductory Survey, 2nd Ed. Academic Press, pp. 483]

The thermodynamic energy equation

First law of Thermodynamics:

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \tag{2.42}$$

where $\dfrac{Dp}{Dt} = \omega$ and J is the diabatic heating rate; the rate of heating per unit mass due to radiation, conduction, and latent heat release.

$$\therefore c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \alpha\omega = J$$

Equation of state: $p\alpha = RT$

$$\therefore \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + \omega\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\omega = \frac{J}{c_p}$$

$$\therefore \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \omega\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T - \omega\left(\frac{RT}{c_p p} - \frac{\partial T}{\partial p}\right) = \frac{J}{c_p}$$

Static stability parameter for the isobaric system: $S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$

$$\therefore \frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T - S_p \omega = \frac{J}{c_p}$$

It can be shown that $S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g}$

For observed lapse rate equal to the dry adiabatic lapse rate, $S_p = 0$

$$\therefore \frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T = \frac{J}{c_p}$$

If the motion is adiabatic, $J = 0$

$$\therefore \frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T = 0$$

Exercise 1: A frontal zone moves over Tshwane overnight so that the local temperature falls at a rate of $1°\text{C} \cdot \text{h}^{-1}$. The wind is blowing from the South at $10\,\text{km} \cdot \text{h}^{-1}$. The temperature is decreasing with latitude at a rate of $10°\text{C}$ per $100\,\text{km}$. Neglecting diabatic heating, and for the case of the observed lapse rate being equal to the dry adiabatic lapse rate, use the thermodynamic energy equation to describe the local rate of temperature change, and the advection of temperature over Tshwane.

Solution:

$$\frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T - S_p \omega = \frac{J}{c_p}$$

$$\therefore \frac{\partial T}{\partial t} = -\overline{V} \cdot \overline{\nabla} T$$

Left hand side: $\frac{\partial T}{\partial t} = -1°\text{C} \cdot \text{h}^{-1}$

Right hand side:

$$\begin{aligned}
-\overline{V} \cdot \overline{\nabla} T &= -(u\bar{i} + v\bar{j}) \cdot \left(\frac{\partial T}{\partial x}\bar{i} + \frac{\partial T}{\partial y}\bar{j} \right) \\
&= -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} \\
&= -v\frac{\partial T}{\partial y} \qquad \text{(since } u = 0) \\
&= -(10\,\text{km} \cdot \text{h}^{-1}) \left(\frac{-10°\text{C}}{100\,\text{km}} \right) \qquad (v > 0) \\
&= 1°\text{C} \cdot \text{h}^{-1}
\end{aligned}$$

In order for the left and right hand sides to be equal, the right-hand side must be reduced by $2°\text{C} \cdot \text{h}^{-1}$. Such a reduction may be caused by adiabatic cooling due to vertical advection.

$$\begin{aligned} \therefore \text{Right hand side} &= 1°\text{C} \cdot \text{h}^{-1} - 2°\text{C} \cdot \text{h}^{-1} \\ &= -1°\text{C} \cdot \text{h}^{-1} \\ &= \text{Left hand side} \end{aligned}$$

Exercise 2: Explain in words what this form of the thermodynamic energy equations represents:

$$\frac{\partial T}{\partial t} + \overline{V} \cdot \overline{\nabla} T = 0$$

Solution: It represents the balance between the local rate of temperature change and the advection of temperature.

Balanced flow

Assumptions:

1. flows are steady state (i.e. time independent)
2. no vertical component of velocity

Natural coordinates

Defined by the orthogonal set of unit vectors $\bar{t}$, $\overline{n}$ and $\overline{k}$

$\bar{t}$: parallel to the horizontal velocity at each point

$\overline{n}$: normal to the horizontal velocity; positive to the left of the flow direction

$\overline{k}$: vertically upward

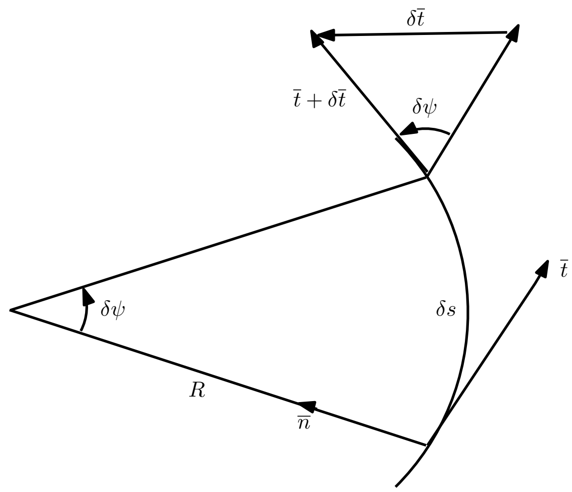
Horizontal velocity $\overline{V} = V\bar{t}$; where V is the horizontal speed, non-negative scalar

$$V \equiv \frac{Ds}{Dt}$$

where $s(x, y, t)$ is the distance along the curve of parcel.

Acceleration following the motion:

$$\begin{aligned}\frac{D\overline{V}}{Dt} &= \frac{D(V\bar{t})}{Dt} \\ &= \frac{DV}{Dt}\bar{t} + \frac{D\bar{t}}{Dt}V\end{aligned}$$



According to the figure: $\delta s = |R|\delta\psi$ (“$s = r\theta$”),

where R is the radius of curvature following the parcel motion.

$$\delta\psi = \frac{\delta s}{|R|} = \frac{\delta\bar{t}}{|\bar{t}|} \qquad \text{(considered small triangle)}$$

$$|\bar{t}| = 1$$

$$\Longrightarrow \frac{\delta s}{|R|} = |\delta\bar{t}|$$

$$\frac{|\delta\bar{t}|}{\delta s} = \frac{1}{|R|}$$

In the limit $\delta s \to 0$, $\delta\bar{t}$ becomes parallel to $\overline{n}$

$$\Longrightarrow \frac{D\bar{t}}{Ds} = \frac{\overline{n}}{R} \qquad \text{(because } \overline{n} \text{ is a unit vector: } |\overline{n}| = 1)$$

$$\frac{D\bar{t}}{Dt} = \frac{D\bar{t}}{Ds}\frac{Ds}{Dt} = \frac{\overline{n}}{R}V \qquad \left(V \equiv \frac{Ds}{Dt}\right)$$

$$\Longrightarrow \frac{D\overline{V}}{Dt} = \bar{t}\frac{DV}{Dt} + V\left(\frac{\overline{n}}{R}V\right)$$

$$\frac{D\overline{V}}{Dt} = \bar{t}\frac{DV}{Dt} + \overline{n}\frac{V^2}{R} \tag{3.8}$$

where:

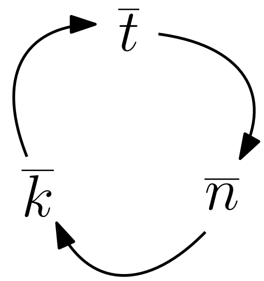
$\frac{D\overline{V}}{Dt}$ is the acceleration following the motion.

$\bar{t}\frac{DV}{Dt}$ is the rate of change of speed of the air parcel.

$\overline{n}\frac{V^2}{R}$ is the centripetal acceleration due to curvature of trajectory.

Acceleration due to **Coriolis force**:

$$
\begin{aligned}
&= -f\overline{k} \times \overline{V} \\
&= -f\overline{k} \times V\overline{t} \\
&= -fV\overline{n}
\end{aligned}
$$



Horizontal pressure gradient $= -\overline{\nabla}_p\Phi$

In natural coordinate system: $= -\overline{\nabla}_p\Phi = -\left(\bar{t}\frac{\partial\Phi}{\partial s} + \overline{n}\frac{\partial\Phi}{\partial n}\right)$

Since $\frac{D\overline{V}}{Dt} = \underbrace{-f\overline{k} \times \overline{V} - \overline{\nabla}_p\Phi}_{(3.2)} = -fV\overline{n} - \left(\bar{t}\frac{\partial\Phi}{\partial s} + \overline{n}\frac{\partial\Phi}{\partial n}\right)$

And $\frac{D\overline{V}}{Dt} = \bar{t}\frac{DV}{Dt} + \overline{n}\frac{V^2}{R}$

$\underbrace{\frac{DV}{Dt} = -\frac{\partial\Phi}{\partial s}}_{(3.9)}$ and $\frac{V^2}{R} = -fV - \frac{\partial\Phi}{\partial n} \implies \underbrace{\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n}}_{(3.10)}$

$\frac{DV}{Dt} = -\frac{\partial\Phi}{\partial s}$: force balance **parallel** to the direction of flow.

$\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n}$: force balance **normal** to the direction of flow.

For motion parallel to geopotential height then Φ remains unchanged:

$$
\begin{aligned}
&\frac{\partial\Phi}{\partial s} = 0 \\
\therefore\ &\frac{DV}{Dt} = 0 \\
&\implies \text{Speed is constant following the motion}
\end{aligned}
$$

If the geopotential gradient normal to the direction of motion is constant along a trajectory

$$\frac{\partial \Phi}{\partial n} = 0$$
$$\therefore \frac{V^2}{R} + fV = 0$$
$$\therefore R = \frac{-V^2}{fV} = \frac{-V}{f}$$
$$\Longrightarrow \text{radius of curvature, } R\text{, is constant.}$$

Geostrophic flow

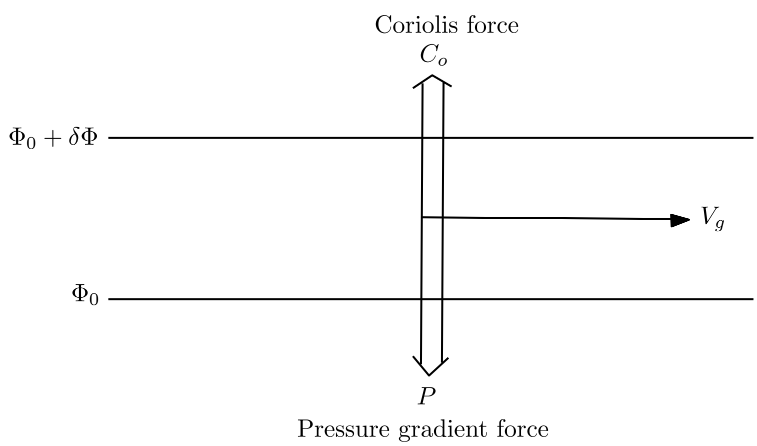
Geostrophic motion: flow in a straight line $(R \to \pm\infty)$ parallel to height contours.

For $R \to \pm\infty, \dfrac{V^2}{R} \to 0$

In geostrophic motion the horizontal components of the Coriolis force and pressure gradient force are in **exact balance**, thus $V = V_g$

$$\therefore 0 + fV = fV_g = -\frac{\partial \Phi}{\partial n} \tag{3.11}$$

The balance



The actual wind can be in exact geostrophic motion only if the height contours are parallel to latitude circles.

Although the geostrophic wind is generally a good approximation to the actual wind in extra-tropical synoptic-scale disturbances, in some special cases this is not true!

Inertial flow

If the geopotential field is uniform on an isobaric surface so that the horizontal pressure gradient vanishes $\left(\dfrac{\partial \Phi}{\partial n} = 0\right)$:

$$\frac{V^2}{R} + fV = 0 \tag{3.12}$$

(3.12): Coriolis force and centrifugal force are balanced.

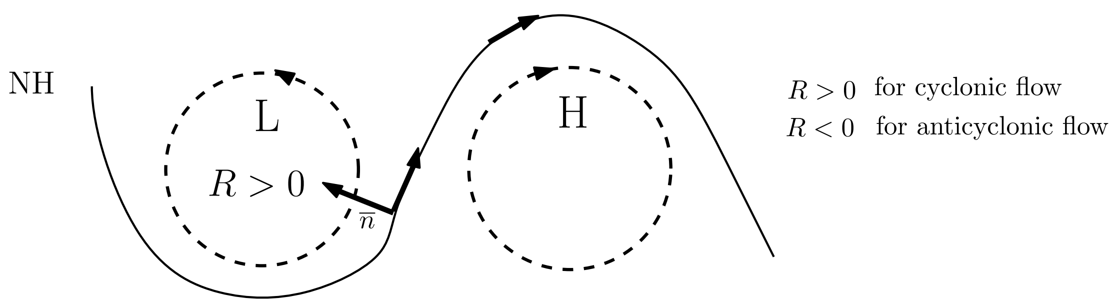
$$R = -\frac{V}{f}$$

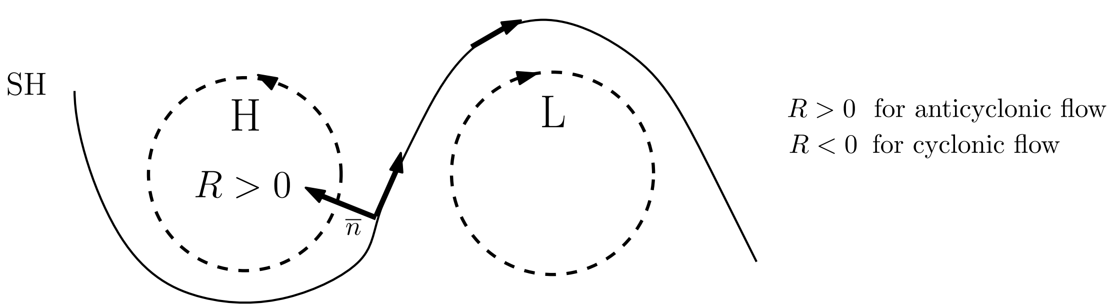
NOTE:

1) In **the atmosphere** motions are nearly always generated and maintained by pressure gradient forces
2) The condition of uniform pressure required for pure inertial flow **rarely exist**

Balanced flow

Consider the natural coordinate system where the unit vector $\overline{n}$ is normal to the horizontal velocity and is positive to the left of the flow direction. This configuration applies to both hemispheres.





$R > 0$ when the centre of curvature is in the positive $\overline{n}$ direction.

Regarding $\frac{\partial \Phi}{\partial n}$:

For the NH:

For LOW pressure system

$$\begin{aligned} \delta\Phi &< 0 \qquad \text{(geopotential decreases towards centre)} \\ \delta n &> 0 \qquad (\overline{n} \text{ pointing towards centre of low}) \\ \therefore \frac{\partial \Phi}{\partial n} &< 0 \end{aligned}$$

For HIGH pressure system

$$\begin{aligned} \delta\Phi &> 0 \qquad \text{(geopotential increases towards centre)} \\ \delta n &< 0 \qquad (\overline{n} \text{ pointing towards centre of low}) \\ \therefore \frac{\partial \Phi}{\partial n} &< 0 \end{aligned}$$

For the SH:

For LOW pressure system $\delta\Phi < 0, \delta n < 0$ therefore $\frac{\partial \Phi}{\partial n} > 0$

For HIGH pressure system $\delta\Phi > 0, \delta n > 0$ therefore $\frac{\partial \Phi}{\partial n} > 0$

Cyclostrophic flow

If the horizontal scale of an atmospheric disturbance is small enough, the Coriolis force may be neglected when compared with the centrifugal force and the pressure gradient force:

Centrifugal force $\frac{V^2}{R} \gg fV$

Pressure gradient force $\frac{\partial \Phi}{\partial n} \gg fV$

From Eq. (3.10):

$$\begin{aligned} \frac{V^2}{R} &= -\frac{\partial \Phi}{\partial n} \\ V &= \left(-R\frac{\partial \Phi}{\partial n}\right)^{1/2}, \qquad \text{the cyclostrophic wind speed.} \end{aligned}$$

Case 1: $R > 0$ and $\frac{\partial \Phi}{\partial n} > 0$

$$\begin{aligned} R\frac{\partial \Phi}{\partial n} &> 0 \\ \therefore -R\frac{\partial \Phi}{\partial n} &< 0 \\ \implies \text{Negative root, } V &\text{ physically impossible} \end{aligned}$$

Case 2: $R < 0$ and $\dfrac{\partial \Phi}{\partial n} > 0$

$$R\frac{\partial \Phi}{\partial n} < 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} > 0$$
$$\implies \text{Positive root, } V \text{ physically possible}$$

Case 3: $R > 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

$$R\frac{\partial \Phi}{\partial n} < 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} > 0$$
$$\implies \text{Positive root, } V \text{ physically possible}$$

Case 4: $R < 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

$$R\frac{\partial \Phi}{\partial n} > 0$$
$$\therefore -R\frac{\partial \Phi}{\partial n} < 0$$
$$\implies \text{Negative root, } V \text{ physically impossible}$$

The mathematically positive roots of the speed of the cyclostrophic wind correspond to only two physically possible solutions:

$$R < 0 \text{ and } \frac{\partial \Phi}{\partial n} > 0 \quad \text{(Case 2)}$$

and

$$R > 0 \text{ and } \frac{\partial \Phi}{\partial n} < 0 \quad \text{(Case 3)}$$

Consider the figures on the next page. Since the Coriolis force is not a factor, around lows, cyclostrophic winds can turn either clockwise or counterclockwise. As discussed in the balanced flow section, $\overline{n}$ is positive to the left of the flow direction, $R > 0$ when curvature centre is in $\overline{n}$ direction.

Therefore,

NH: Cyclonic flow $R > 0$ and anti-cyclonic flow $R < 0$

SH: Cyclonic flow $R < 0$ and anti-cyclonic flow $R > 0$

Regarding $\dfrac{\partial \Phi}{\partial n}$:

Since we are dealing here with (intense) low pressure systems $\delta\Phi < 0$ for cyclonic and anti-cyclonic flow, and for both hemispheres.

NH: Cyclonic flow $\delta n > 0$ therefore $\dfrac{\partial \Phi}{\partial n} < 0$

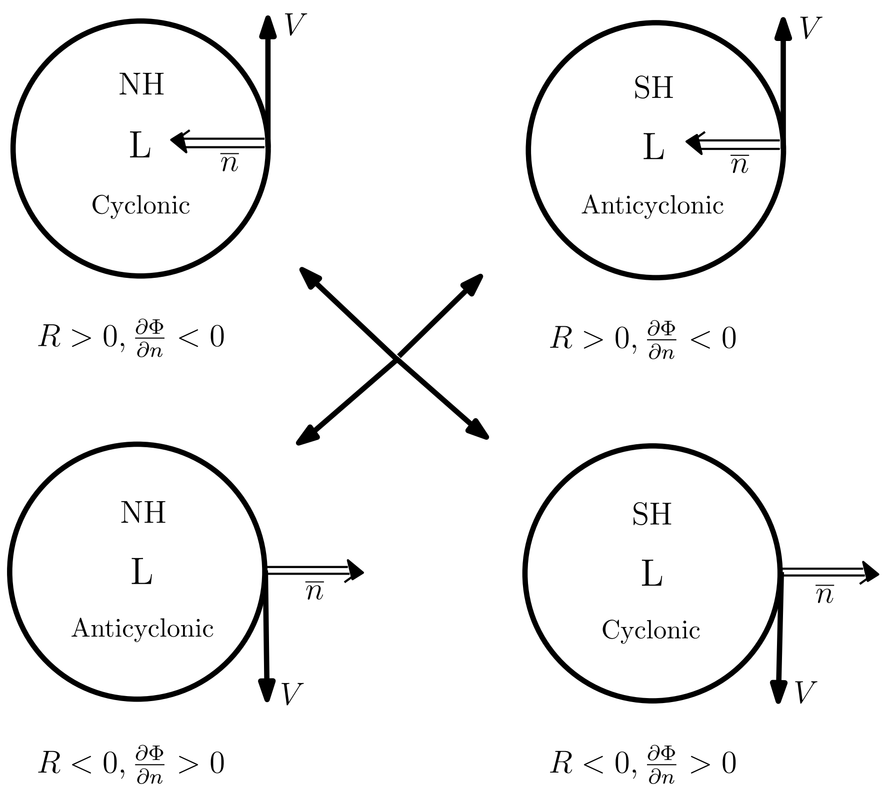
SH: Cyclonic flow $\delta n < 0$ therefore $\dfrac{\partial \Phi}{\partial n} > 0$

NH: anti-cyclonic flow $\delta n < 0$ therefore $\dfrac{\partial \Phi}{\partial n} > 0$

SH: anti-cyclonic flow $\delta n > 0$ therefore $\dfrac{\partial \Phi}{\partial n} < 0$

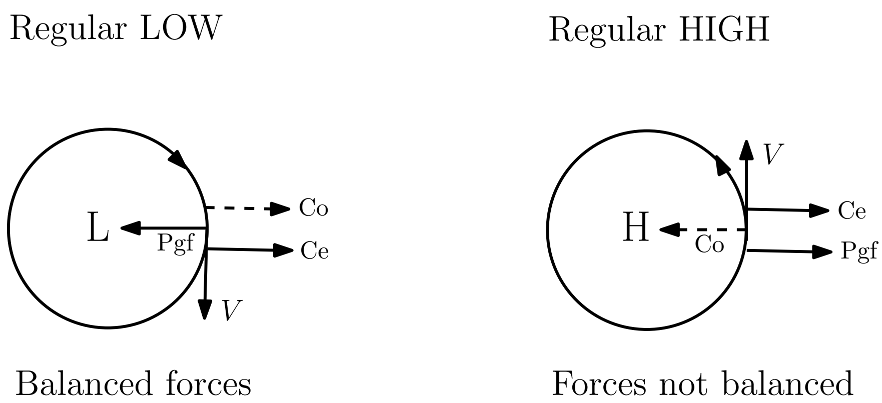
Cyclostrophic wind classification:

$\dfrac{\partial \Phi}{\partial n}$ $(+/-)$	$R > 0$	$R < 0$
Positive	Case 1: unphysical	Case 2: physical NH: anti-cyclonic; SH: cyclonic
Negative	Case 3: physical NH: cyclonic; SH: anti-cyclonic	Case 4: unphysical



Exercise: By means of drawing circular symmetric motion figures, explain why there can be no cyclostrophic balance around a small high pressure centre.

Solution:



For cyclostrophic flow, the Coriolis force becomes negligible compared with other two forces. For a LOW, there can still be balance. In this case between Ce and Pgf. However, for a HIGH, Ce and Pgf point in the same direction. So no balance is possible here.

The gradient wind approximation

Gradient Flow: Horizontal frictionless flows that is parallel to the height contours so that the tangential acceleration vanishes, i.e. $\dfrac{DV}{Dt} = 0$.

Gradient Flow is a 3-way balance among:

1) The Coriolis force
2) The centrifugal force
3) The horizontal pressure gradient force

A **gradient wind** is just the wind component parallel to the height contour that satisfies:

$$\underbrace{\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}}_{\text{the gradient wind equation}} \tag{3.10}$$

For a quadratic equation $ax^2 + bx + c = 0$, solving the equation for x

$$x = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

Therefore, solving for V in Eq. (3.10), $a = \frac{1}{R}$, $b = f$, $c = \frac{\partial \Phi}{\partial n}$

$$V = \frac{-f \pm \left(f^2 - 4\frac{1}{R}\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}}}{\frac{2}{R}}$$
$$= -\frac{fR}{2} \pm \frac{R}{2}\left(f^2 - \frac{4}{R}\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}}$$
$$= -\frac{fR}{2} \pm \left(\left(\frac{R}{2}\right)^2 \left(f^2 - \frac{4}{R}\frac{\partial \Phi}{\partial n}\right)\right)^{\frac{1}{2}}$$

For geostrophic flow

$$fV_g = -\frac{\partial \Phi}{\partial n}$$
$$\therefore \frac{\partial \Phi}{\partial n} = -fV_g$$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} \tag{3.15}$$

Determining the **mathematically possible roots** of (3.15)

$$\underbrace{V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}}}_{\text{the \textbf{gradient wind}}}$$

By the geostrophic approximation $V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$ i.t.o the **pressure gradient**

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial \Phi}{\partial n}\right)^{\frac{1}{2}} \tag{A}$$

Objective: Determine the cases for which the solution of (A) is both positive and real

The gradient wind approximation: SOUTHERN HEMISPHERE $(f < 0)$

Case 1: For $R > 0$ and $\frac{\partial \Phi}{\partial n} > 0$

$$fR < 0, \quad -\frac{fR}{2} > 0$$
$$R\frac{\partial \Phi}{\partial n} > 0, \quad -R\frac{\partial \Phi}{\partial n} < 0$$

Since V has to be real, $\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} > 0$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{\frac{1}{2}}$$

For **positive** root:

$$V = \text{Positive value}\left(-\frac{fR}{2}\right) + \text{positive}(+\surd)$$

$$\therefore V > 0, \text{ and therefore physically possible.}$$

For **negative** root: Consider $\frac{f^2R^2}{4} > 0$

$$\therefore \frac{f^2R^2}{4} > \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} < 0\right)$$

$$\therefore \pm\left(\frac{f^2R^2}{4}\right)^{1/2} > \pm\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering negative roots:

$$-\left(\frac{f^2R^2}{4}\right)^{1/2} < -\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} < -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V > -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} - \underbrace{\left|\frac{fR}{2}\right|}_{\frac{fR}{2} < 0 \text{ from } *} = -\frac{fR}{2} - \left(-\frac{fR}{2}\right)$$

$\therefore V > 0,$ and therefore physically possible.

$$* \quad \boxed{\begin{aligned} |x| &= x \quad \text{for} \quad x > 0 \\ |x| &= -x \quad \text{for} \quad x < 0 \end{aligned}}$$

Case 2: For $R < 0$ and $\frac{\partial\Phi}{\partial n} > 0$

$$fR > 0, \; -\frac{fR}{2} < 0$$

$$R\frac{\partial\Phi}{\partial n} < 0, \ \ -R\frac{\partial\Phi}{\partial n} > 0$$

For real V, $\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} > 0$

For **negative** root:

$$V = -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$= \text{negative value} - \text{positive value}$$

$$\therefore V < 0, \text{and therefore physically impossible.}$$

For **positive** root: $\frac{f^2R^2}{4} > 0$

$$\frac{f^2R^2}{4} < \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} > 0\right)$$

$$\therefore \pm\left(\frac{f^2R^2}{4}\right)^{1/2} < \pm\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering positive roots:

$$+\left(\frac{f^2R^2}{4}\right)^{1/2} < +\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} + \left(\frac{f^2R^2}{4}\right)^{1/2} < -\frac{fR}{2} + \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V > -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} + \left|\frac{fR}{2}\right| = -\frac{fR}{2} + \frac{fR}{2}$$

$$\therefore V > 0, \text{and therefore physically possible.}$$

Case 3: For $R > 0$ and $\frac{\partial\Phi}{\partial n} < 0$

$$fR < 0, \ \ -\frac{fR}{2} > 0$$

$$R\frac{\partial\Phi}{\partial n} < 0, \ \ -R\frac{\partial\Phi}{\partial n} > 0$$

Since V has to be real, $\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} > 0$

For **positive** root:

$$V = \text{positive value}\left(-\frac{fR}{2}\right) + \text{positive value}(+\surd)$$

$$\therefore V > 0, \text{and therefore physically possible.}$$

For **negative** root: $\dfrac{f^2R^2}{4} > 0$

$$\therefore \frac{f^2R^2}{4} < \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} > 0\right)$$

$$\therefore \pm\left(\frac{f^2R^2}{4}\right)^{1/2} < \pm\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering negative roots:

$$-\left(\frac{f^2R^2}{4}\right)^{1/2} > -\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} > -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V < -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} - \left|\frac{fR}{2}\right| = -\frac{fR}{2} - \left(-\frac{fR}{2}\right) \qquad \left(\text{since } \frac{fR}{2} < 0\right)$$

$\therefore V < 0$, and therefore physically impossible.

Case 4: For $R < 0$ and $\dfrac{\partial\Phi}{\partial n} < 0$

$$fR > 0, \ \ -\frac{fR}{2} < 0$$

$$R\frac{\partial\Phi}{\partial n} > 0, \ \ -R\frac{\partial\Phi}{\partial n} < 0$$

For real V, $\dfrac{f^2R^2}{4} - R\dfrac{\partial\Phi}{\partial n} > 0$

For **negative** root:

$$V = -\frac{fR}{2} - \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$= \text{negative value} - \text{positive value}$$

$\therefore V < 0$, and therefore physically impossible.

For **positive** root: $\dfrac{f^2R^2}{4} > 0$

$$\frac{f^2R^2}{4} > \frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} \qquad \left(\text{since} - R\frac{\partial\Phi}{\partial n} < 0\right)$$

$$\therefore \pm\left(\frac{f^2R^2}{4}\right)^{1/2} > \pm\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

But we are considering positive roots:

$$+\left(\frac{f^2R^2}{4}\right)^{1/2} > +\left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$\therefore -\frac{fR}{2} + \left(\frac{f^2R^2}{4}\right)^{1/2} > -\frac{fR}{2} + \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2} = V$$

$$\therefore V < -\frac{fR}{2} - \left(\frac{f^2R^2}{4}\right)^{1/2} = -\frac{fR}{2} + \left|\frac{fR}{2}\right| = -\frac{fR}{2} + \frac{fR}{2}$$

$\therefore V < 0,$ and therefore physically impossible.

The following table is a summary of the four cases

Gradient wind classification in the Southern Hemisphere

$\frac{\partial\Phi}{\partial n}$ $(+/-)$	$R > 0$ (anti-cyclonic)	$R < 0$ (cyclonic)
Positive	Case 1 $+$ root : physical $-$ root : physical	Case 2 $+$ root : physical $-$ root : unphysical
Negative	Case 3 $+$ root : physical $-$ root : unphysical	Case 4 $+$ root : unphysical $-$ root : unphysical

For **cyclonic** flow ($R < 0$) the only physically possible configuration is:

$$R < 0 \text{ and } \frac{\partial\Phi}{\partial n} > 0 \qquad \text{(similar to result in the cyclostrophic wind classification section)}$$

$\frac{\partial\Phi}{\partial n} > 0$ makes sense since $\delta\Phi < 0$ and $\delta n < 0$. Also consider the geostrophic wind equation

$$V_g = -\frac{1}{f}\frac{\partial\Phi}{\partial n} \tag{3.11}$$

$$\therefore V_g = -\frac{1}{\text{(neg)}}\text{(pos)} > 0$$

For **anti-cyclonic** flow ($R > 0$), $\frac{\partial\Phi}{\partial n}$ can be either positive or negative.

So, for $\frac{\partial\Phi}{\partial n} > 0$: $V_g = -\frac{1}{\text{(neg)}}\text{(pos)} > 0$ as was found for cyclonic flow.

However, for $\frac{\partial\Phi}{\partial n} < 0$: $V_g = -\frac{1}{\text{(neg)}}\text{(neg)} < 0$

Again consider:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \tag{3.10}$$

The geostrophic wind is defined by

$$fV_g = -\frac{\partial \Phi}{\partial n} \tag{3.11}$$

$$\begin{aligned} \therefore \frac{V^2}{R} + fV &= fV_g \\ \frac{1}{fV}\left(\frac{V^2}{R} + fV\right) &= \frac{1}{fV}(fV_g) \\ \frac{V}{fR} + 1 &= \frac{V_g}{V}, \text{the ratio of the geostrophic wind to the gradient wind.} \end{aligned}$$

From the balanced flow section:

Northern Hemisphere ($f > 0$):

$$\begin{aligned} R > 0 &\text{ for cyclonic flow} \\ R < 0 &\text{ for anti-cyclonic flow} \\ \therefore Rf > 0 &\text{ for cyclonic flow} \\ Rf < 0 &\text{ for anti-cyclonic flow} \end{aligned}$$

Southern Hemisphere ($f < 0$):

$$\begin{aligned} R < 0 &\text{ for cyclonic flow} \\ R > 0 &\text{ for anti-cyclonic flow} \\ \therefore Rf > 0 &\text{ for cyclonic flow} \\ Rf < 0 &\text{ for anti-cyclonic flow} \end{aligned}$$

Typical values for V,f and R: $5\,\mathrm{m\cdot s^{-1}}$, $10^{-4}\,\mathrm{s^{-1}}$ and $500\,\mathrm{km}$.

$$\therefore \frac{V}{fR} = \frac{5\,\mathrm{m\cdot s^{-1}}}{10^{-4}\,\mathrm{s^{-1}}\,500000\,\mathrm{m}} = 0.1$$

For cyclonic flow (both hemispheres):

$$\begin{aligned} \frac{V_g}{V} &= 1 + 0.1 = 1.1 \\ \therefore V_g &= 1.1 \times V \\ \implies V_g &> V \end{aligned}$$

For anti-cyclonic flow (both hemispheres):

$$\frac{V_g}{V} = 1 - 0.1 = 0.9$$
$$\therefore V_g = 0.9 \times V$$
$$\implies V_g < V$$

Therefore, the geostrophic wind is an overestimate of the balanced wind in a region of cyclonic curvature, and an underestimate in a region of anti-cyclonic curvature.

Next we want to illustrate the force balances for the permitted solutions. For this purpose we want to determine

1) the direction of these forces

2) their relative sizes

Take note of the following:

1) the centrifugal force (C_e) always points outwards,

2) the pressure gradient force (Pgf) is always from a high to a low pressure system,

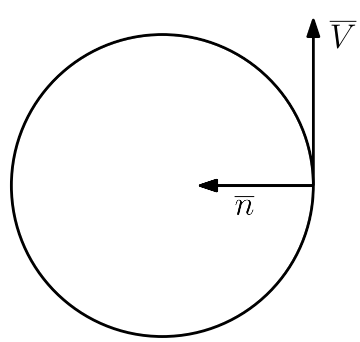
3) the Coriolis force (C_o) is to the left of the motion in the Southern Hemisphere, and

4) the vector $\overline{n}$ is positive to the left and perpendicular to $\overline{V}$.

First, determine the sign of R: $R > 0 \implies$ anti-cyclone, or $R < 0 \implies$ cyclone.

Second, draw a circular flow structure with $\overline{n}$ perpendicular and to the left of $\overline{V}$.

Example, for $R > 0$ (anti-cyclone):

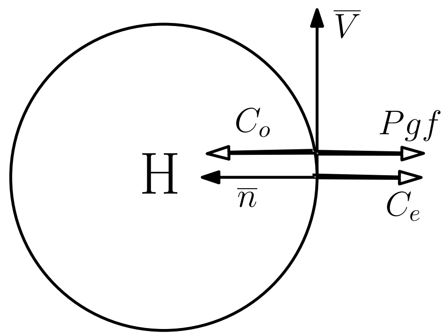


Third, consider the sign of $\dfrac{\partial \Phi}{\partial n}$

For example, for $\dfrac{\partial \Phi}{\partial n} > 0$:

For the circular flow above, $\delta n > 0$ in the direction towards the centre of the circle. Therefore $\delta \Phi > 0$ in the direction of the centre of the circle, which means that here we are dealing with a high pressure system.

Now we can complete the circular flow structure above by including the appropriate forces:



This flow represents Case 1 $\left(R > 0, \dfrac{\partial \Phi}{\partial n} > 0\right)$

Take note: For now we are not concerned with the relative sizes of the vectors that represent the three forces, but only with the direction of these forces. The discussion of the relative sizes is still to follow.

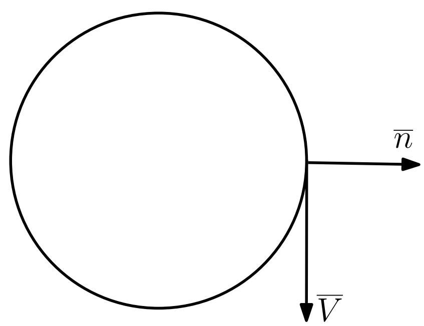
C_o : Coriolis force

C_e : Centrifugal force

Pgf : Pressure gradient force

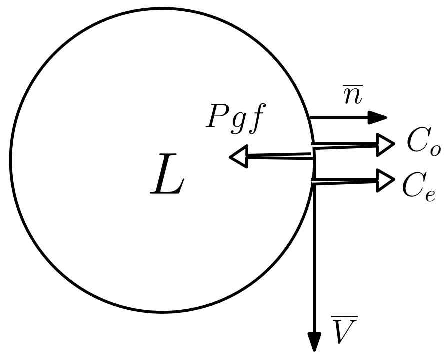
For Case 2, $R < 0$ and $\dfrac{\partial \Phi}{\partial n} > 0$

Owing to the sign of R (< 0), the circular flow will be cyclonic:



In the direction towards centre of circle, $\delta n < 0$ and $\dfrac{\delta \Phi}{\delta n} > 0$, therefore $\delta \Phi < 0$, which means we are dealing with a low pressure system.

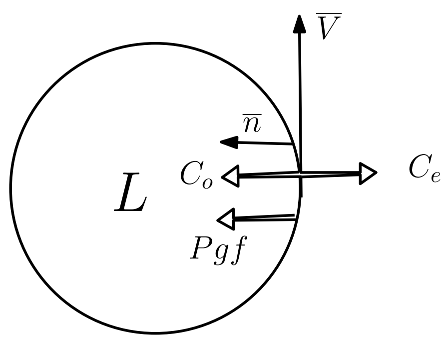
Now complete the circular flow structure by including the appropriate forces:



For Case 3, $R > 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

Since $R > 0$, the circular flow is similar to Case 1 (anti-cyclonic) and $\delta n > 0$.

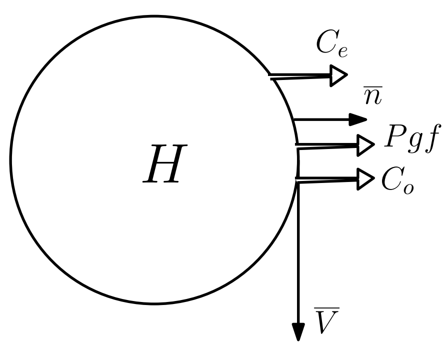
However $\dfrac{\delta \Phi}{\delta n} < 0$, therefore $\delta \Phi < 0$, which means we are dealing with a low pressure system that rotates anti-cyclonically.



For Case 4, $R < 0$ and $\dfrac{\partial \Phi}{\partial n} < 0$

Since $R < 0$, the circular flow is similar to Case 2 (cyclonic) and $\delta n < 0$.

However $\dfrac{\delta \Phi}{\delta n} < 0$, therefore $\delta \Phi > 0$, which means we are dealing with a high pressure system that rotates cyclonically.



This configuration leads to an unbalanced circular flow structure (all the forces are in the same direction) that is not physically possible.

We have now managed to determine the direction of the three forces for various circular flow structures. Next, we will attempt to obtain insight into the relative sizes of the forces associated with the flow structures.

Consider the following parameter values representative of extra-tropical circulation in the Southern Hemisphere:

$$\begin{aligned} f &= -10^{-4}\,\mathrm{s}^{-1} \\ |R| &= 10^{6}\,\mathrm{m} \\ \left|\frac{\partial \Phi}{\partial n}\right| &= 10^{-3}\,\mathrm{m\,s}^{-2} \end{aligned}$$

Calculate the gradient wind speeds by using (3.15) of Holton 4 for each of the four cases.

<u>Case 1</u>: $R > 0$ and $\dfrac{\delta \Phi}{\delta n} > 0$

Remember that for this case, V has two physical solutions, one associated with the a positive root of (3.15) and one with a negative root.

The first term on the right of (3.15) is $-\dfrac{fR}{2}$. Since for this case $-\dfrac{fR}{2} > 0$,

$$\begin{aligned} V &> -\frac{fR}{2} \text{ for a positive root} \\ V &< -\frac{fR}{2} \text{ for a negative root} \end{aligned}$$

By using the parameters above

$$\begin{aligned} -\frac{fR}{2} &= \frac{-\left(-10^{-4}\,\mathrm{s}^{-1}\right)\left(+10^{6}\,\mathrm{m}\right)}{2} \\ &= 50\,\mathrm{m\,s}^{-1} \end{aligned}$$

$$\begin{aligned} \therefore V &= 50\,\mathrm{m\,s}^{-1} \pm \left(\frac{\left(-10^{-4}\,\mathrm{s}^{-1}\right)^2\left(+10^{6}\,\mathrm{m}\right)^2}{4} - \left(+10^{6}\,\mathrm{m}\right)\left(10^{-3}\,\mathrm{m\,s}^{-2}\right)\right)^{\frac{1}{2}} \\ &= 50\,\mathrm{m\,s}^{-1} \pm 38.73\,\mathrm{m\,s}^{-1} \end{aligned}$$

For a positive root, $V = 50 + 38.73 = 88.73\,\mathrm{m\,s}^{-1} > 50$, an anomalous high and confirms that $V > -\dfrac{fR}{2}$

This very large speed associated with a positive root makes the circular anti-cyclonic flow <u>anomalous</u>. The high speed of the gradient wind will result in the centrifugal and Coriolis force becoming large.

To demonstrate this point, consider (3.10): $\dfrac{V^2}{R} + fV = -\dfrac{\partial \Phi}{\partial n}$

$$\frac{V^2}{R} = \text{centripetal acceleration or centrifugal force per unit mass } (C_e)$$
$$fV = \text{Coriolis force per unit mass } (C_o)$$
$$\frac{\partial \Phi}{\partial n} = \text{horizontal pressure gradient force (per unit mass; } Pgf)$$

Take note that centripetal force and centrifugal force are the exact same force, just in opposite directions because they are experienced from different forms of reference (i.e, inward vs. outward).

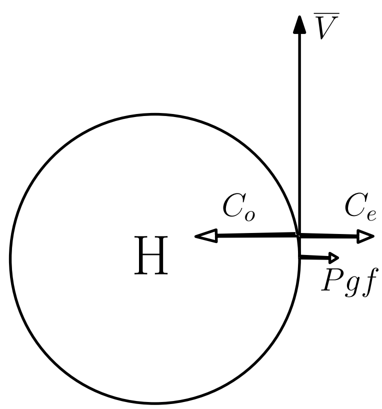
Next we will calculate the absolute strength of these forces by using the parameter values presented above.

For Case 1's positive root:

$$C_e = \frac{\left(88.73\,\mathrm{m\,s^{-1}}\right)^2}{10^6\,\mathrm{m}} = 7.873 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}}$$
$$C_o = \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(88.73\,\mathrm{m\,s^{-1}}\right) = 8.873 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}}$$
$$\frac{\partial \Phi}{\partial n} = 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}$$

Therefore, by using representative numbers we demonstrated that for Case 1 and positive roots, the centrifugal and Coriolis forces are similar in strength, and about one order of magnitude stronger than the pressure gradient force.

The circular flow structure for an anomalous high may take the following form:

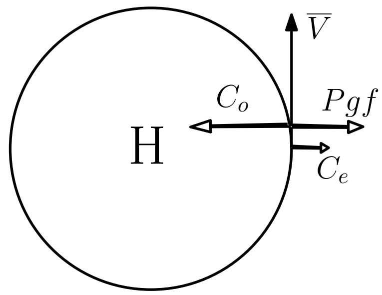


For Case 1's negative root, $V = 50 - 38.73 = 11.27\,\mathrm{m\,s^{-1}}$, a regular high and confirms that $V < -\frac{fR}{2}$

$$C_e = \frac{\left(11.27\,\mathrm{m\,s^{-1}}\right)^2}{10^6\,\mathrm{m}} = 1.270 \times 10^{-4}\,\mathrm{m\,s^{-2}} \sim 10^{-4}\,\mathrm{m\,s^{-2}}$$
$$C_o = \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(11.27\,\mathrm{m\,s^{-1}}\right) = 1.127 \times 10^{-3}\,\mathrm{m\,s^{-2}} \sim 10^{-3}\,\mathrm{m\,s^{-2}}$$
$$Pgf = 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}$$

The much more realistic gradient wind speed associated with a negative root does not make the centrifugal force large as was the case with the positive root, but results in the Coriolis and pressure gradient forces becoming of similar magnitude.

The circular flow structure for such a regular high may take the following form:



For Case 2, only a positive root is associated with a physical solution for the gradient wind.

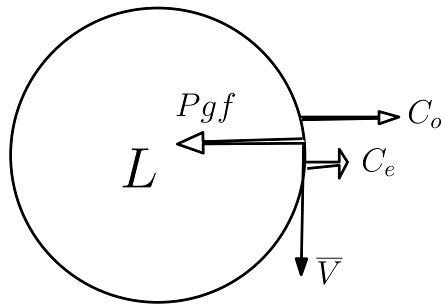
Remember that for this case $R < 0$ (cyclonic) and $\dfrac{\partial \Phi}{\partial n} > 0$

Therefore, $-\dfrac{fR}{2} < 0$ and is equal to $-50\,\mathrm{m\,s^{-1}}$

$$\begin{aligned}
\therefore V &= -50\,\mathrm{m\,s^{-1}} + \left(\frac{\left(-10^{-4}\,\mathrm{s^{-1}}\right)^2 \left(-10^{6}\,\mathrm{m}\right)^2}{4} - \left(-10^{6}\,\mathrm{m}\right)\left(10^{-3}\,\mathrm{m\,s^{-2}}\right) \right)^{\frac{1}{2}} \\
&= -50\,\mathrm{m\,s^{-1}} + 59.16\,\mathrm{m\,s^{-1}} \\
&= 9.16\,\mathrm{m\,s^{-1}}, \text{ a regular low} \\
\therefore V &> -\frac{fR}{2}
\end{aligned}$$

$$\begin{aligned}
C_e &= \frac{\left(9.16\,\mathrm{m\,s^{-1}}\right)^2}{10^{6}\,\mathrm{m}} = 8.390 \times 10^{-5}\,\mathrm{m\,s^{-2}} \sim 10^{-4}\,\mathrm{m\,s^{-2}} \\
C_o &= \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(9.16\,\mathrm{m\,s^{-1}}\right) = 9.16 \times 10^{-4}\,\mathrm{m\,s^{-2}} \sim 10^{-3}\,\mathrm{m\,s^{-2}} \\
Pgf &= 10^{-3}\,\mathrm{m\,s^{-2}}, \text{ given}
\end{aligned}$$

The circular flow structure for a regular low may take the following form:



For Case 3, only a positive root is associated with a physical solution for the gradient wind.

For this case, $R > 0$ (anti-cyclonic) and $\dfrac{\partial \Phi}{\partial n} < 0$

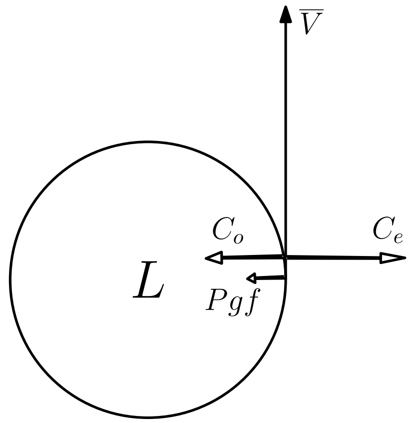
Therefore, $-\dfrac{fR}{2} > 0$ and is equal to $50\,\mathrm{m\,s^{-1}}$

$$
\begin{aligned}
\therefore V &= 50\,\mathrm{m\,s^{-1}} + \left(\frac{\left(-10^{-4}\,\mathrm{s^{-1}}\right)^2\left(+10^{6}\,\mathrm{m}\right)^2}{4} - \left(+10^{6}\,\mathrm{m}\right)\left(-10^{-3}\,\mathrm{m\,s^{-2}}\right)\right)^{\frac{1}{2}} \\
&= 50\,\mathrm{m\,s^{-1}} + 59.16\,\mathrm{m\,s^{-1}} \\
&= 109.16\,\mathrm{m\,s^{-1}} \\
\therefore V &> -\frac{fR}{2}
\end{aligned}
$$

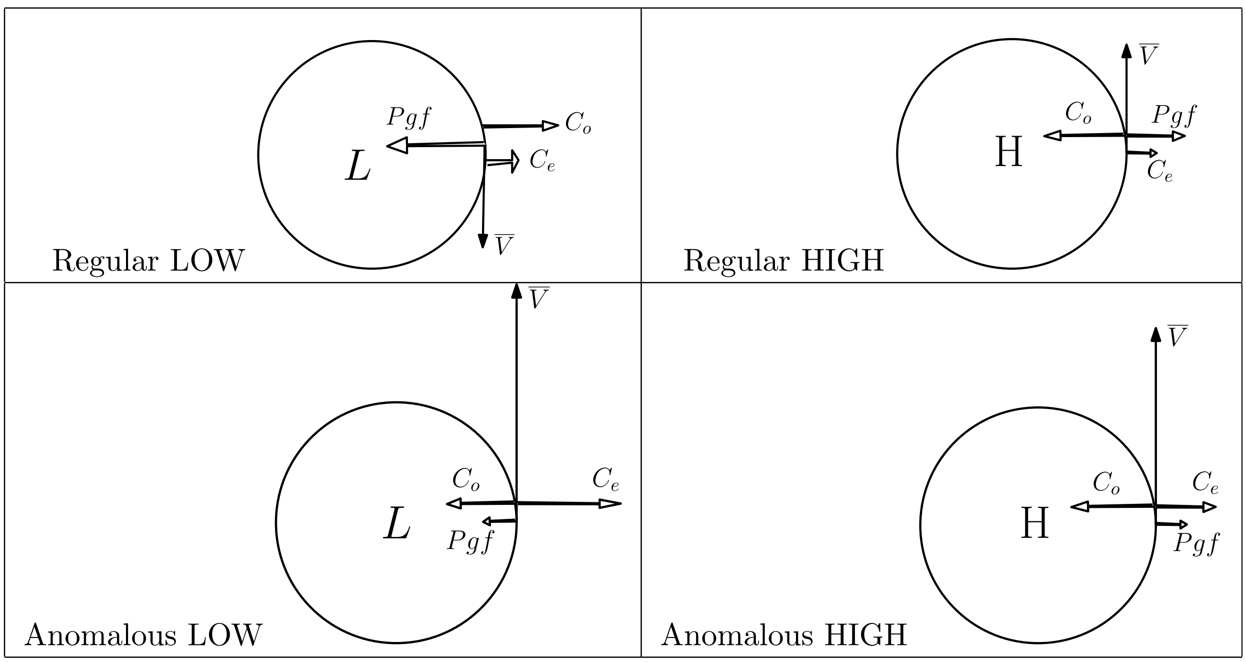
Take note that for this case we are dealing with a low pressure system that rotates counter-clockwise (anti-cyclonically) in the Southern Hemisphere! This flow orientation and high gradient wind speed makes this circular flow an anomalous low.

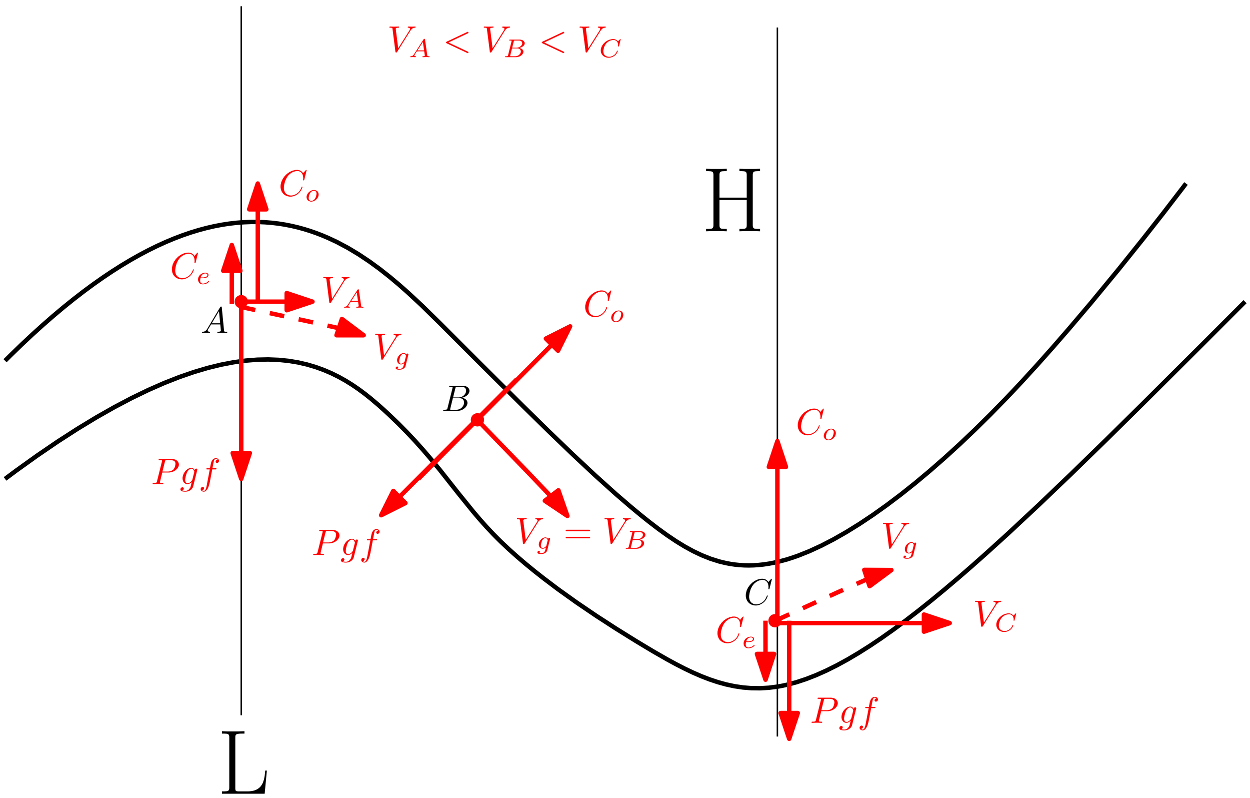
$$
\begin{aligned}
C_e &= \frac{\left(109.16\,\mathrm{m\,s^{-1}}\right)^2}{10^{6}\,\mathrm{m}} = 1.192 \times 10^{-2}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}} \\
C_o &= \left(10^{-4}\,\mathrm{s^{-1}}\right)\left(109.16\,\mathrm{m\,s^{-1}}\right) = 1.092 \times 10^{-2}\,\mathrm{m\,s^{-2}} \sim 10^{-2}\,\mathrm{m\,s^{-2}} \\
Pgf &= 10^{-3}\,\mathrm{m\,s^{-2}},\ \text{given}
\end{aligned}
$$

The circular flow structure for an anomalous low may take the following form:


$\overline{V}$
C_o
C_e
L
Pgf

SUMMARY:


Pgf
C_o
L
C_e
$\overline{V}$
Regular LOW
$\overline{V}$
C_o
Pgf
H
C_e
Regular HIGH
$\overline{V}$
C_o
C_e
L
Pgf
Anomalous LOW
$\overline{V}$
C_o
C_e
H
Pgf
Anomalous HIGH



Consider a highly idealized trough-ridge system in the Southern Hemisphere at an isobaric level, say the level at 500 hPa. We have shown above that for a regular low pressure system the $Pgf > C_o > C_e$, and for a regular high pressure system $C_o > Pgf > C_e$. Moreover, we have already shown that for cyclonic flow $V_g > V$, and for anti-cyclonic flow $V_g < V$. The gradient wind speeds at the locations marked A (on trough line), B (in between trough and ridge lines) and C (ridge line) have gradient wind speeds of respectively V_A, V_B and V_C. Since the gradient wind speed is less than the geostrophic wind speed at the trough and is greater than the geostrophic wind speed at the ridge, we have $V_A < V_B < V_C$. The gradient wind speed therefore increases from the trough line towards the ridge line. However, in the absence of curvature (typically at point B on the idealised flow structure) C_e is absent and therefore perfect geostrophic balance is achieved before the gradient wind accelerates further towards the ridge.

Next we will try to obtain insight into the behaviour of the gradient wind speed for regular high (Case 1, negative root) and regular low (Case 2, positive root) pressure systems when only the pressure gradient force is allowed to increase incrementally (1% increase over 250 iterations), while the other two forces are kept constant. Such an increase in the pressure gradient is reminiscent of a strengthening high pressure system and a deepening low pressure system, respectively. Figure 5 shows the respective gradient wind speeds obtained by using the parameter values presented above. The gradient wind speed of the regular high pressure system becomes imaginary after 93 iterations. This result suggests that for a regular high pressure system, there may be a limit to the intensity of such a developing system. Moreover, gradient wind speeds increase indefinitely for the case of a regular low. This result, on the other hand, suggests that there may not be a limit to the depth of a low pressure centre.

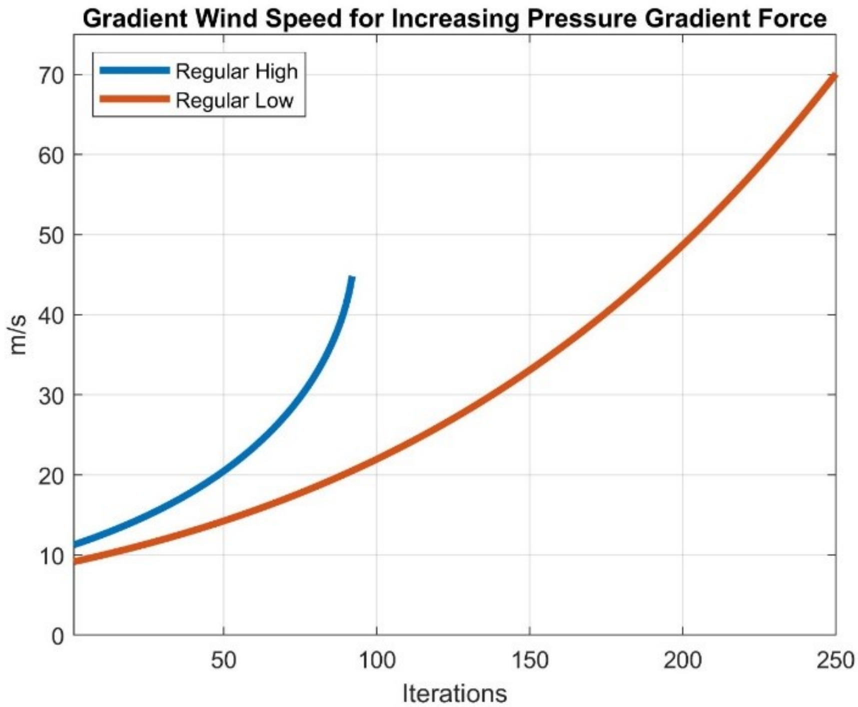


Figure 5: Gradient wind speed of regular high and regular low pressure systems by incrementally increasing the pressure gradient force by 1% for 250 iterations.

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - R\frac{\partial \Phi}{\partial n} \right)^{\frac{1}{2}} \tag{A}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \tag{B}$$

An increase in the Pgf on the RHS of (B) implies that the Coriolis force and Centrifugal force on the LHS of (B) must also increase for the forces to be in balance. For a constant f and R, it follows that V must increase if the Pgf were to increase. A regular high is associated with $R > 0$, $Pgf > 0$ and a negative root in (A). Since the term $-R\frac{\partial \Phi}{\partial n} < 0$ and the term $\frac{f^2R^2}{4} > 0$, it follows that if the Pgf increases substantially whereby $\left|\frac{f^2R^2}{4}\right| < \left|-R\frac{\partial \Phi}{\partial n}\right|$, the term under the root will be negative resulting in V becoming imaginary or undefined. This is depicted in Figure 5.

Figure 6 shows the results when the pressure gradient and Coriolis forces are kept constant, but the radius of curvature, R, is allowed to respectively increase and decrease incrementally (1% change over 250 iterations). The figure shows that there is a limit to how small a high pressure system can become, since the gradient wind becomes imaginary after only 92 iterations of decreasing R. When R is allowed to increase, the gradient wind speed decreases. This result indicates that high pressure systems are generally large and have light winds. High pressure systems with small R and the resulting large gradient wind speeds are unstable and seldom occur in nature. For a low pressure system, Figure 6 shows that the effect of a varying R is likely quite small in increasing the gradient wind speed of this type of weather system. Since $R < 0$ in a regular low, it follows from (B) that the centrifugal force actually counteracts the work of the Coriolis force to balance the Pgf. An intense weather system such as a tropical cyclone therefore relies primarily on the Coriolis force and Pgf to initiate its development. However, once the tropical cyclone has reached maturity, the Pgf and centrifugal forces dominate. The following example illustrates this notion.

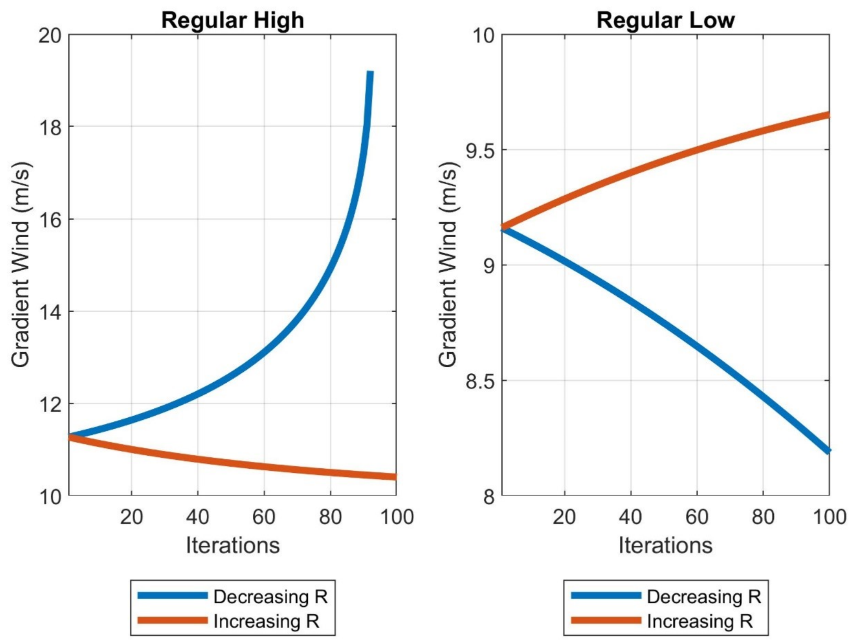


Figure 6: Gradient wind speeds of regular high and regular low pressure systems by incrementally changing the radius of curvature, R.

Consider the case of a strong tropical cyclone in the Southern Hemisphere with a central pressure of $950\,\mathrm{hPa}$ and a maximum wind speed of $60\,\mathrm{m\,s^{-1}}$. Assume that the normal pressure outside the tropical cyclone is $1010\,\mathrm{hPa}$. The centre of the storm is at $15°\mathrm{S}$. To calculate the Pgf of this cyclone requires evaluation of the $\dfrac{\partial \Phi}{\partial n}$ term in the gradient wind equation. Since

$$\begin{aligned}
\delta\Phi = \frac{1}{\rho}\delta p &= \frac{(1010 - 950) \times 10^2\,\mathrm{Pa}}{1.2\,\mathrm{kg\,m^{-3}}} \\
&= 5000\,\mathrm{Pa\,kg^{-1}\,m^3} \\
&= 5000\,\mathrm{N\,m^{-2}\,kg^{-1}\,m^3} \\
&= 5000\,\mathrm{kg\,m\,s^{-2}\,m^{-2}\,kg^{-1}\,m^3} \\
&= 5000\,\mathrm{m^2\,s^{-2}}
\end{aligned}$$

A tropical cyclone can be considered an intermediate size system since its typical length scale of $500\,\mathrm{km}$ is larger than that of a coastal low, but smaller than that of a strong mid-latitude cyclone. Therefore, the length scale of a tropical cyclone is the vortex radius. Therefore $\frac{\delta\Phi}{\delta n} \sim \frac{\delta p}{\rho R}$, where R is the radius of curvature.

$\therefore \frac{\delta\Phi}{\delta n} \sim \frac{5000\,\mathrm{m^2\,s^{-2}}}{500\,000\,\mathrm{m}} = 0.01\,\mathrm{m\,s^{-2}}$, which is an order of magnitude stronger then the value used before.

Since a tropical cyclone always rotates clockwise in the Southern Hemisphere it cannot be an anomalous low as one of the physically possible solutions of the gradient wind equation. Therefore, tropical cyclones are regular low pressure systems with $R < 0$ and $\frac{\delta\Phi}{\delta n} > 0$. In order to estimate this cyclone's radius and the importance of the centrifugal force relative to the Coriolis force, consider the gradient wind equation:

$$\frac{V^2}{R} + fV = -\frac{\partial\Phi}{\partial n}, \qquad R < 0, \frac{\partial\Phi}{\partial n} > 0$$

$$\begin{aligned} f &= 2\Omega\sin(-15^\circ) \\ &= 2\cdot\left(7.292\times10^{-5}\,\mathrm{s^{-1}}\right)\cdot\sin(-15^\circ) \\ &= -3.775\times10^{-5}\,\mathrm{s^{-1}} \end{aligned}$$

$$\begin{aligned} \frac{V^2}{R} &= -\left(\frac{\partial\Phi}{\partial n} + fV\right) \\ \therefore R &= -V^2\left(\frac{\partial\Phi}{\partial n} + fV\right)^{-1} \\ &= -60^2\left(10^{-2} + \left(-3.775\times10^{-5}\right)(60)\right)^{-1} \\ &= -4.654\times10^{5}\,\mathrm{m} \\ &= -465.4\,\mathrm{km}, \text{ a realistic length scale!} \end{aligned}$$

$$\text{Centrifugal force:} \quad \begin{aligned} \frac{V^2}{R} &= \frac{60^2(\mathrm{m\,s^{-1}})^2}{-4.654\times10^5\,\mathrm{m}} \\ &= -7.735\times10^{-3}\,\mathrm{m\,s^{-2}} \end{aligned}$$

$$\text{Coriolis force:} \quad \begin{aligned} fV &= -3.775\times10^{-5}(60) \\ &= -2.265\times10^{-3}\,\mathrm{m\,s^{-2}} \end{aligned}$$

Therefore, $\frac{C_e}{C_o} = 3.4$. This result is in contrast to the previous calculation of the relative sizes of the three forces where, for a regular low, the Coriolis force is larger than the centrifugal force. Notwithstanding, for a tropical cyclone presented here, the centrifugal force dominates the Coriolis force by a factor of 3. However, such a cyclone is still not cyclostrophic for which the centrifugal force is expected to be an order of magnitude larger than the Coriolis force.

Exercise 1: Consider an anti-cyclone and the case of positive pressure gradient forces. At a radius of $100\,\mathrm{km}$ and associated geostrophic wind speed of $2.4\,\mathrm{m\,s^{-1}}$, calculate the gradient wind speeds. Is the ratio between the given geostrophic wind and the calculated gradient winds in agreement with your result found above? Next, redo the gradient wind calculation, but this time double the geostrophic wind speed and interpret this result. The Coriolis parameter is $-10^{-4}\,\mathrm{s^{-1}}$.

Solution: For anti-cyclone: $R > 0$ and (given) $\dfrac{\partial\Phi}{\partial n} > 0$

$$\begin{aligned} R &= 100\,000\,\mathrm{m} \\ V_g &= 2.4\,\mathrm{m\,s^{-1}} \\ f &= -10^{-4}\,\mathrm{s^{-1}} \end{aligned}$$

$$\begin{aligned} V &= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} \\ &= -\frac{(-10^{-4})(100\,000)}{2} \pm \left(\frac{(-10^{-4})^2(100\,000)^2}{4} + (-10^{-4})(100\,000)(2.4)\right)^{\frac{1}{2}} \\ &= 5 \pm (25 - 24)^{\frac{1}{2}} \\ &= 5 \pm 1\,\mathrm{m\,s^{-1}} \end{aligned}$$

For positive root: $V = 6\,\mathrm{m\,s^{-1}}$

For negative root: $V = 4\,\mathrm{m\,s^{-1}}$

For anti-cyclonic flow, it has been demonstrated that:

$$V_g < V$$

Since both V solutions are greater than $2.4\,\mathrm{m\,s^{-1}}$, the given geostrophic wind speed. Therefore, the ratio between the geostrophic wind and the gradient wind is in agreement with the result.

For double geostrophic wind, $V_g = 4.8\,\mathrm{m\,s^{-1}}$

$$\begin{aligned} V &= 5 \pm (25 - 48)^{\frac{1}{2}} \\ &= 5 \pm (\text{negative value})^{\frac{1}{2}} \end{aligned}$$

Therefore, $4.8\,\mathrm{m\,s^{-1}}$ as a geostrophic wind is unrealistically high since this leads to an unphysical solution for V.

Exercise 2: When the two terms under the square root of the solved quadratic equation (3.10) are perfectly balanced (their sum equals zero), determine the ratio of the anti-cyclonic gradient wind speed to the geostrophic wind speed for the same pressure gradient.

Solution:

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}}$$

$$\text{Given : } \quad \frac{f^2R^2}{4} + fRV_g = 0 \implies V = -\frac{fR}{2}$$
$$\therefore \frac{fR}{4} + V_g = 0$$
$$\therefore -\frac{2V}{4} + V_g = 0$$
$$\therefore V = 2V_g$$
$$\therefore \frac{V}{V_g} = 2$$

Exercise 3: Show that as the pressure gradient approaches zero the gradient wind reduces to the geostrophic wind for a normal anti-cyclone [Hint: make use of this approximation: when variable x approaches zero, the square root of $1 + x$ is equal to $1 + x/2$].

Solution: Since the pressure gradient approaches zero, so does V_g because $V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$.

The following approximation has been given: when $x \to 0$, $(1 + x)^{1/2} = 1 + \frac{x}{2}$

We therefore consider the square root term of the gradient wind equation:

$$\pm\left(\frac{f^2R^2}{4} + fRV_g\right)^{\frac{1}{2}} = \pm\left[\frac{f^2R^2}{4}\left(1 + \frac{4V_g}{fR}\right)\right]^{\frac{1}{2}}$$
$$= \pm\frac{fR}{2}\left(1 + \frac{4V_g}{fR}\right)^{\frac{1}{2}}$$
$$= \pm\frac{fR}{2}\left(1 + \frac{1}{2}\cdot\frac{4V_g}{fR}\right)$$

$$\therefore V = -\frac{fR}{2} \pm \frac{fR}{2}\left(1 + \frac{2V_g}{fR}\right)$$

For this case, we are only interested in the positive root.

$$\therefore V = -\frac{fR}{2} + \frac{fR}{2} + V_g$$
$$\therefore V = V_g$$

An alternative view on balanced flow

The gradient wind equation can be expressed in terms of the geostrophic wind:

$$\frac{V^2}{R} + fV - fV_g = 0,\ f < 0 \text{ in the Southern Hemisphere}$$
$$\implies \text{Centrifugal force} + \text{Coriolis force} + \text{pressure gradient force are in balance.}$$

The radius of curvature, R, can be obtained from this equation:

$$\frac{V^2}{R} = f(V_g - V)$$
$$\therefore R = \frac{V^2}{f(V_g - V)}$$

The balance of forces equation divided by the Coriolis force, fV, leads to

$$\frac{V^2}{fVR} + 1 - \frac{fV_g}{fV} = 0$$
$$\therefore \frac{V}{fR} + 1 - \frac{V_g}{V} = 0$$
$$\implies \frac{V}{fR} = \frac{V_g}{V} - 1$$

The last equation represents a straight line of the form $y = mx + c$, with $m = 1$ and $c = -1$: $y = x - 1$, with $y = \frac{V}{fR}$ and $x = \frac{V_g}{V}$.

Consider the following values of x and then calculate the corresponding y-values:

$$x \in \left\{-1, 0, \frac{1}{2}, 1, 2, 3\right\}.$$

For $x = -1$:

$$\frac{V_g}{V} = -1$$
$$\therefore V = -V_g,\ \text{defined here as anti-geostrophic flow.}$$
$$y = x - 1 = -1 - 1 = -2$$
$$R = \frac{V^2}{f(-V - V)} = -\frac{V}{2f} > 0,\ \text{anti-cyclonic flow}$$

For $x = 0$:

$$\frac{V_g}{V} = 0$$
$$\therefore V_g = 0$$
$$y = 0 - 1 = -1$$
$$R = \frac{V^2}{f(0 - V)} = -\frac{V}{f} > 0,\ \text{anti-cyclonic flow}$$

For $x = \frac{1}{2}$:

$$
\begin{aligned}
\frac{V_g}{V} &= \frac{1}{2} \\
\therefore V &= 2V_g \\
\therefore V &> V_g, \text{ anti-cyclonic flow} \\
y &= \frac{1}{2} - 1 = -\frac{1}{2} \\
R &= \frac{V^2}{f\left(\frac{1}{2}V - V\right)} = -\frac{2V}{f} > 0, \text{ anti-cyclonic flow}
\end{aligned}
$$

For $x = 1$:

$$
\begin{aligned}
\frac{V_g}{V} &= 1 \\
\therefore V &= V_g \\
y &= 1 - 1 = 0 \\
R &= \frac{V^2}{f(V - V)} \text{ is undefined.}
\end{aligned}
$$

For $x = 2$:

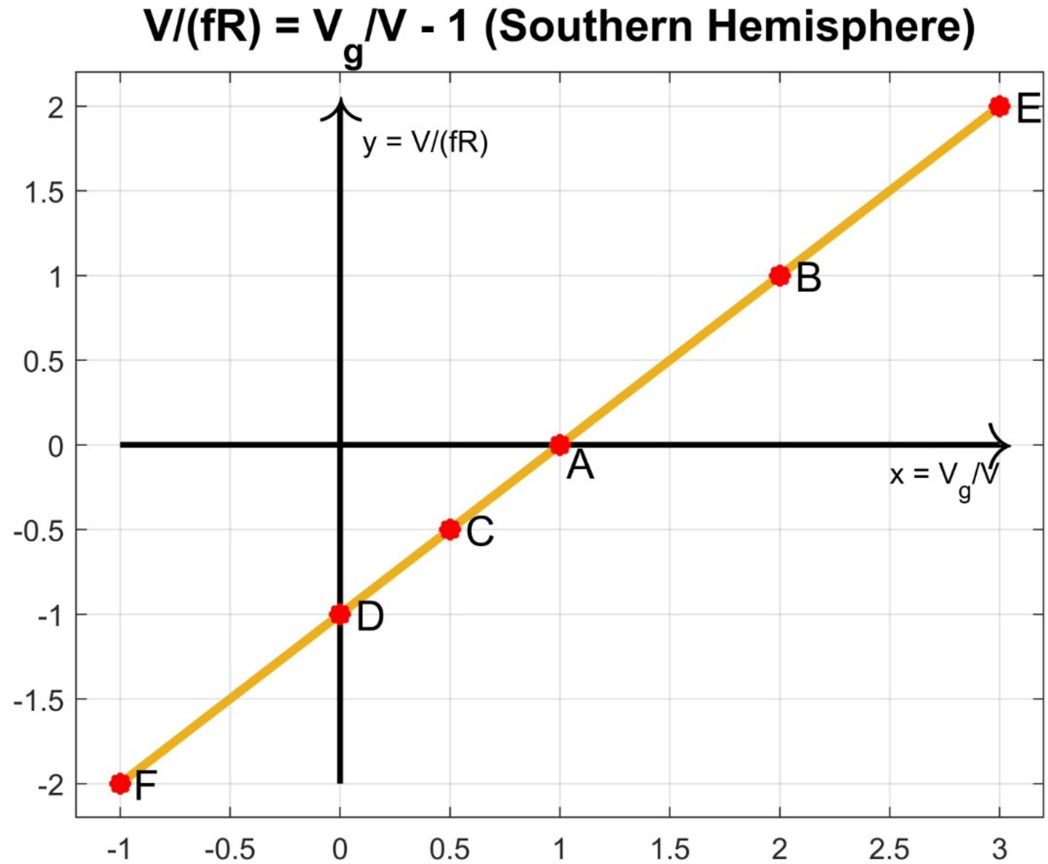
$$
\begin{aligned}
\frac{V_g}{V} &= 2 \\
\therefore V &= \frac{1}{2}V_g \\
\therefore V &< V_g, \text{ cyclonic flow} \\
y &= 2 - 1 = 1 \\
R &= \frac{V^2}{f\,(2V - V)} = \frac{V}{f} < 0, \text{ cyclonic flow}
\end{aligned}
$$

For $x = 3$:

$$
\begin{aligned}
\frac{V_g}{V} &= 3 \\
\therefore V &= \frac{1}{3}V_g \\
\therefore V &< V_g, \text{ cyclonic flow} \\
y &= 3 - 1 = 2 \\
R &= \frac{V^2}{f\,(3V - V)} = \frac{V}{2f} < 0, \text{ cyclonic flow}
\end{aligned}
$$

Also, $f = \frac{V}{2R} < \frac{V}{R}$ obtained from $x = 2$. Therefore, with x increasing, f decreases resulting in the Coriolis force becoming smaller.

We have now calculated the y-values of the straight line, resulting in the figure below:



For the x-value cases above, where $y < 0$, the flow has been found to be anti-cyclonic $\left(\text{e.g., } x = -1, 0, \frac{1}{2}\right)$. Conversely, where $y > 0$, the flow is cyclonic (e.g., $x = 2, 3$). Defining the flow to be sub-geostrophic where $V < V_g$ and super-geostrophic where $V > V_g$, sub-geostrophic flow is found where $x > 1$ and super-geostrophic for $0 < x < 1$. For $x < 0$, the flow is considered to be anti-geostrophic since $V = -V_g$.

Near point A ($x = 1, y = 0$), trajectories are nearly straight since $V \simeq V_g$ and $\frac{V_g}{V} \simeq 1$. However, since $\frac{V}{fR} = \frac{V_g}{V} - 1$, $\frac{V}{fR} \simeq 0$. Moreover, $\frac{V}{fR}$ is a Rossby number with R, the radius of curvature, the length scale. The smaller the Rossby number, the more dominant the Coriolis acceleration in the dynamics becomes, resulting in the Coriolis force and the Pgf becoming approximately balanced. So geostrophic balance is a good approximation near point A.

At point B ($x = 2, y = 1$), $R = \frac{V}{f}$, therefore $f = \frac{V}{R}$

$$\frac{V^2}{R} + fV - fV_g = 0 \text{ becomes } \frac{V^2}{R} + \frac{V}{R}V - fV_g = 0$$

Therefore, the centrifugal and Coriolis forces are equal and together balance the Pgf.

At point C $\left(x = \frac{1}{2}, y = -\frac{1}{2}\right)$, $R = -\frac{2V}{f}$ and $V_g = \frac{1}{2}V$.

$$\frac{V^2}{R} + fV - fV_g = 0 \text{ becomes } V^2\left(-\frac{f}{2V}\right) + fV - f\left(\frac{1}{2}V\right) = 0$$

$$\therefore -\frac{fV}{2} + fV - \frac{fV}{2} = 0$$

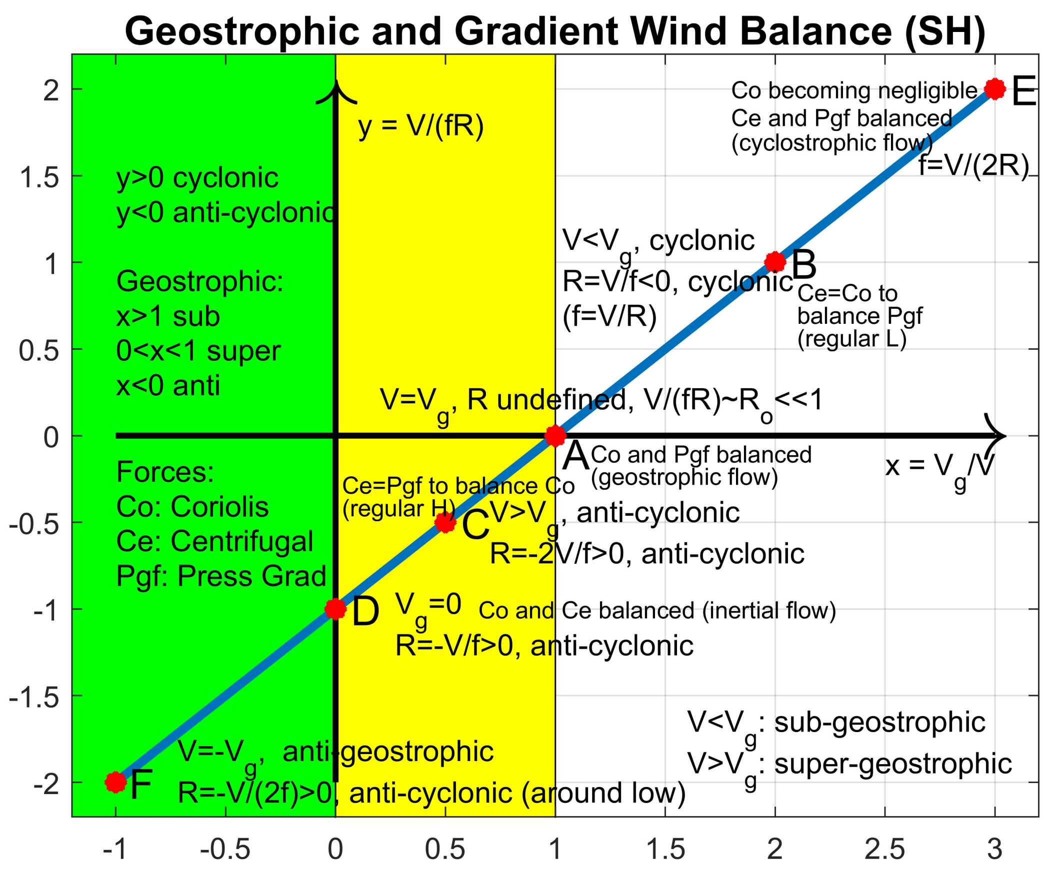
Here the centrifugal force and Pgf are equal and together balance the Coriolis force.

At point D ($x = 0, y = -1$), $V_g = 0$, therefore the $Pgf(= -fV_g)$ is zero, and the Coriolis and centrifugal forces must balance.

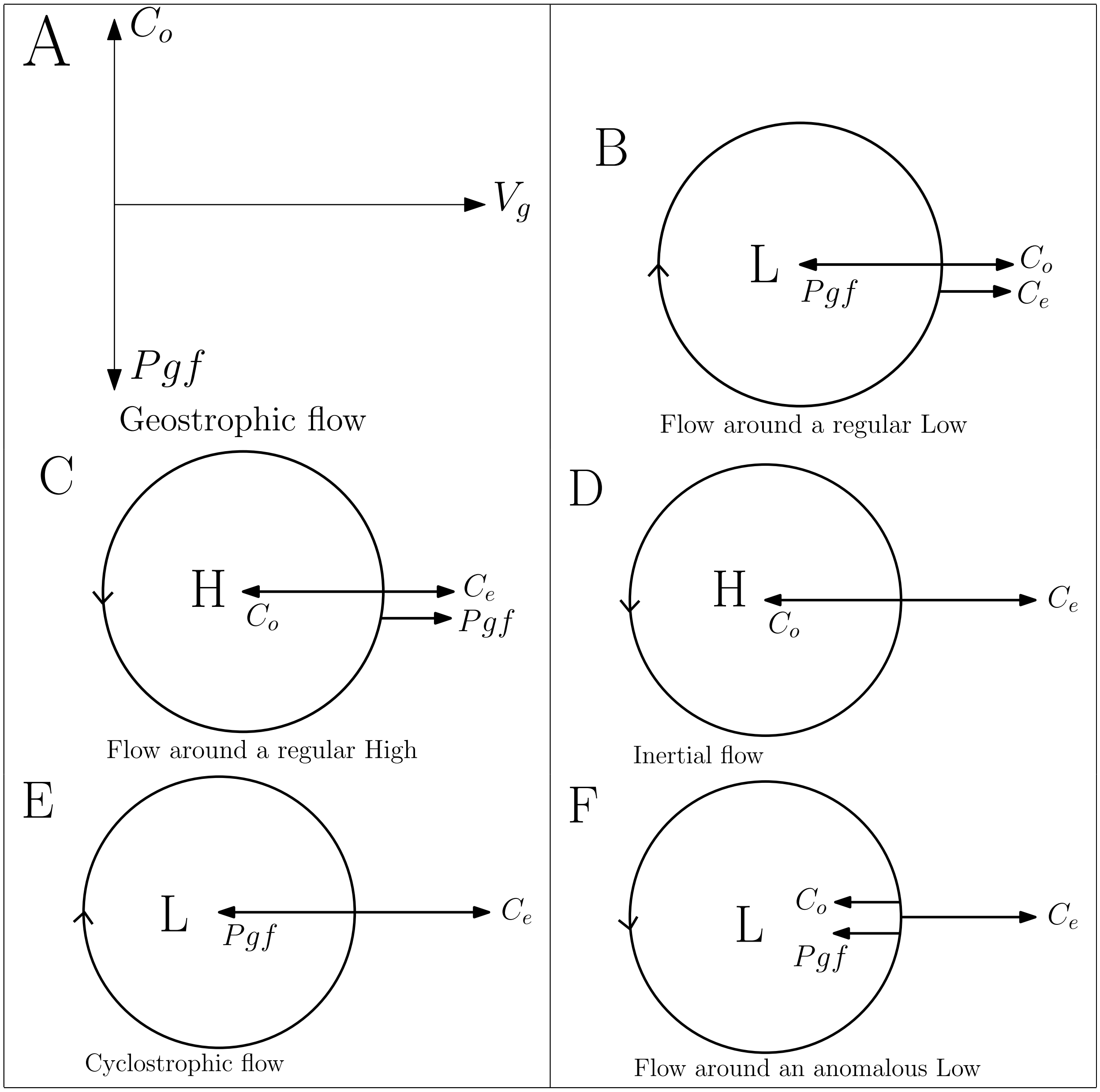
At point E ($x = 3, y = 2$) and with an increasing x-value, the Coriolis force becomes negligible compared with the centrifugal force and Pgf. For this scenario, the centrifugal force must balance the Pgf alone, which is the definition of cyclostrophic balance.

Over the area on the graph where $x < 0$ (including point F), the flow is anti-geostrophic ($V = -V_g$) which means that the flow has anti-cyclonic curvature around a low-pressure system, i.e. an anomalous low, which is never observed.

The figure below summarizes these findings:



A summary of the force balances at each of the points (A-F) is given below:



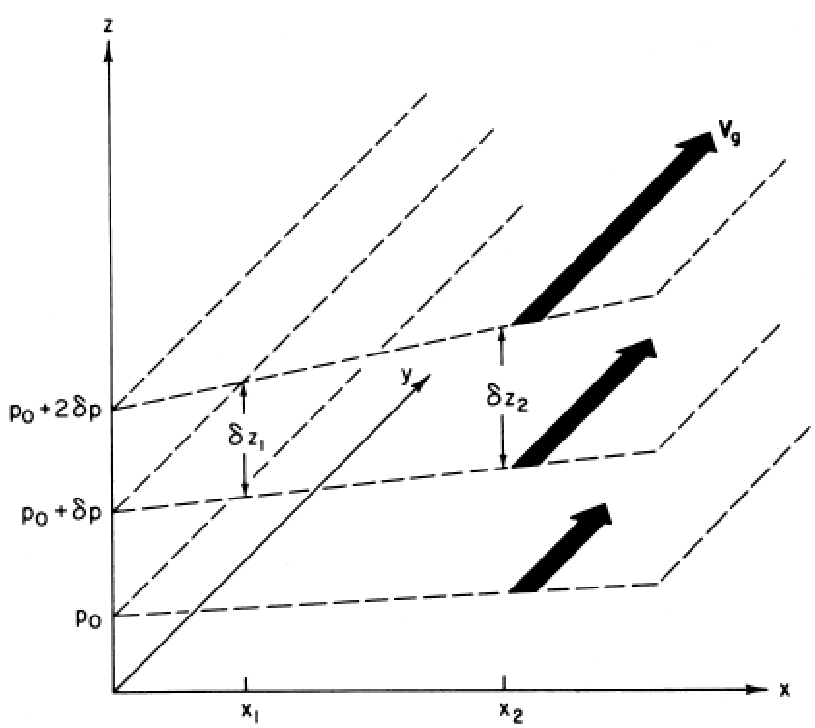
The thermal wind

Isobaric coordinate form of the geostrophic relationship:

$$f\overline{V_g} = \overline{k} \times \overline{\nabla}_p \Phi \qquad \Phi : \text{geopotential}$$

$\Longrightarrow$ geostrophic wind $\propto$ geopotential gradient.

$\Longrightarrow$ geostrophic wind directed along the positive y-axis that increases in magnitude with height requires that the slope of the isobaric surfaces with respect to the x-axis must increase with height.



Hypsometric equation $\Phi(z_2) - \Phi(z_1) = g(z_2 - z_1) = R\int_{p_2}^{p_1} T d\ln p$

For thickness δz corresponding to a pressure interval δp

$$\delta z \approx -\frac{1}{g} RT\delta \ln p$$

$\Longrightarrow$ thickness of the layer between isobaric surfaces $\propto$ temperature of the layer:

$$T(\delta z_1) < T(\delta z_2)$$

$\Longrightarrow$ increase of height of a positive x-directed pressure gradient is associated with a positive x-directed temperature gradient

$\Longrightarrow$ the air in a vertical column at x_2, because it is warmer (less dense), must occupy a greater depth for a given pressure drop than the air at x_1

From $\overline{V_g} = \frac{1}{f}\overline{k} \times \overline{\nabla}_p \Phi$, in isobaric coordinates:

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x} \text{ and } u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$

Equation of state for an ideal gas: $p\alpha = RT$ or $p = \rho RT$ (1.17)
where $\alpha = \rho^{-1}$

Geopotential: $\delta \Phi = g\delta z \implies \delta z = \frac{1}{g}\delta \Phi$

Hydrostatic equation: $\frac{\delta p}{\delta z} = -\rho g \implies \frac{\delta z}{\delta p} = -\frac{1}{\rho g}$

$$\therefore \frac{1}{g}\frac{\delta \Phi}{\delta p} = -\frac{\alpha}{g}$$

$$\lim_{\delta p \to 0} \frac{\delta \Phi}{\delta p} = \frac{\partial \Phi}{\partial p} = -\alpha$$

$$= -\frac{RT}{p} \tag{3.27}$$

$$\therefore T = -\frac{p}{R}\frac{\partial \Phi}{\partial p}$$

Differentiate geostrophic wind components with respect to pressure:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial p}\right) = \frac{1}{f}\frac{\partial}{\partial x}\left(-\frac{RT}{p}\right)$$

$$\therefore p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p$$

$$\therefore \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right) = -\frac{1}{f}\frac{\partial}{\partial y}\left(-\frac{RT}{p}\right)$$

$$\therefore \frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \tag{3.29}$$

As a vector:

$$\underbrace{\frac{\partial \overline{V}_g}{\partial \ln p} = -\frac{R}{f}\overline{k} \times \overline{\nabla}_p T}_{\text{the thermal wind equation}} \tag{3.30}$$

$$\overline{V}_T = -\frac{R}{f}\int_{p_0}^{p_1} \overline{k} \times \overline{\nabla}_p T d \ln p \tag{3.31}$$

$\langle T \rangle$ is the mean temperature in the layer between p_0 and p_1.

$$\begin{aligned} u_T &= +\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \int_{p_0}^{p_1} d \ln p \\ &= -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p [-\ln(p_1) + \ln(p_0)] \\ &= -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right) && (3.32) \\ \text{and } v_T &= -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)_p \int_{p_0}^{p_1} d \ln p \\ &= \frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right) && (3.32) \end{aligned}$$

<u>**Thermal wind**</u> is the vector difference between geostrophic winds at two levels:

$$\begin{aligned} \overline{V}_T &\equiv \overline{V}_g(p_1) - \overline{V}_g(p_0) \qquad (p_1 < p_0) \\ \text{also } u_T = u_{g_1} - u_{g_0} &= -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1) - \left(-\frac{1}{f}\frac{\partial}{\partial y}(\Phi_0)\right) \\ &= -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) && (3.33) \\ v_T = v_{g_1} - v_{g_0} &= \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1) - \frac{1}{f}\frac{\partial}{\partial x}(\Phi_0) \\ &= \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) && (3.33) \\ \Longrightarrow -\frac{R}{f}\left(\frac{\partial \langle T \rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right) &= -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \\ \Longrightarrow R \ln\left(\frac{p_0}{p_1}\right) \int \frac{\partial \langle T \rangle}{\partial y} dy &= \int \frac{\partial}{\partial y}(\Phi_1 - \Phi_0) dy \\ \Longrightarrow \Phi_1 - \Phi_0 &= R \langle T \rangle \ln\left(\frac{p_0}{p_1}\right) && (3.34) \end{aligned}$$

Per definition: $\Phi_1 - \Phi_0 \equiv Z_T g$, where Z_T is the **thickness**

Also

$$\frac{R}{f}\left(\frac{\partial \langle T\rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0)$$

$$R\ln\left(\frac{p_0}{p_1}\right)\int \frac{\partial \langle T\rangle}{\partial x}dx = \int \frac{\partial}{\partial x}(\Phi_1 - \Phi_0)dx$$

$$\implies \Phi_1 - \Phi_0 = R\langle T\rangle \ln\left(\frac{p_0}{p_1}\right) \tag{3.34}$$

The thickness is therefore proportional to the mean temperature in the layer.

$\implies$ lines of **equal thickness** are equivalent to the **isotherms** of mean temperature in the layer.

(3.35):

$$\overline{V}_T = \underbrace{\frac{1}{f}\overline{k} \times \overline{\nabla}(\Phi_1 - \Phi_0)}_{\text{From (3.33)}} = \underbrace{\frac{g}{f}\overline{k} \times \overline{\nabla} Z_T}_{\text{From (3.33),(3.34)}} = \underbrace{\frac{R}{f}\overline{k} \times \overline{\nabla}\langle T\rangle \ln\left(\frac{p_0}{p_1}\right)}_{\text{From (3.32)}}$$

Exercise 1: The mean temperature in the layer between 750 and 500 hPa **decreases eastward** by 2°C per 100 km. If the 700 hPa geostrophic wind is from the southeast at $20\,\mathrm{m\,s^{-1}}$, what is the geostrophic wind speed at 500 hPa? Let $f = -10^{-4}\,\mathrm{s^{-1}}$ [Hint: remember Pythagoras when calculating the geostrophic wind components].

Solution: The mean temperature decreases eastward, so there is no north–south component: $\frac{\partial \langle T\rangle}{\partial y} = 0$ and $\frac{\partial \langle T\rangle}{\partial x} < 0$

$$\therefore u_T = 0$$

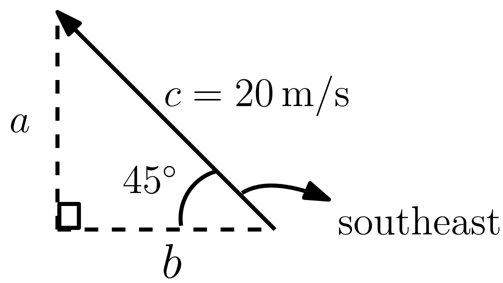
$$v_T = \frac{R}{f}\left(\frac{\partial \langle T\rangle}{\partial x}\right)\ln\left(\frac{p_0}{p_1}\right)$$

where R is the gas constant for dry air.

$$\begin{aligned} R &= 287\,\mathrm{J\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{N\,m\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{kg\,m\,s^{-2}\,m\,K^{-1}\,kg^{-1}} \\ &= 287\,\mathrm{m^2\,s^{-2}\,K^{-1}} \end{aligned}$$

$$\begin{aligned} \therefore v_T &= \frac{287\,\mathrm{m^2\,s^{-2}\,K^{-1}}}{-10^{-4}\,\mathrm{s^{-1}}}\left(-\frac{2\,\mathrm{K}}{100\,000\,\mathrm{m}}\right)\ln\left(\frac{750}{500}\right) \\ &= 23.27\,\mathrm{m\,s^{-1}} \end{aligned}$$

At 750 hPa:



$$c^2 = a^2 + b^2 = 2a^2 \implies a = \left(\frac{20^2}{2}\right)^{1/2} = b$$
$$= 14.14\,\mathrm{m\,s^{-1}}$$

$$u_T = u_{g_1} - u_{g_0} = u_g(500\,\mathrm{hPa}) - u_g(750\,\mathrm{hPa})$$
$$\therefore 0 = u_g(500) - (-14.14)$$
$$\therefore u_g(500) = -14.14\,\mathrm{m\,s^{-1}}$$

$$v_T = v_{g_1} - v_{g_0} = v_g(500\,\mathrm{hPa}) - v_g(750\,\mathrm{hPa})$$
$$\therefore 23.27 = v_g(500) - 14.14$$
$$\therefore v_g(500) = 37.41\,\mathrm{m\,s^{-1}}$$

$$\therefore \overline{V}_g(500) = (-14.14, 37.41)$$

Therefore, geostrophic wind speed at $500\,\mathrm{hPa} = \left|\overline{V}_g(500)\right|$
$$= \left((-14.14)^2 + (37.41)^2\right)^{1/2}$$
$$= 39.99\,\mathrm{m\,s^{-1}}$$

Exercise 2: Consider the values in the previous exercise (Exercise 1) above, what is the mean temperature advection in the 750 to 500 hPa layer?

Solution: Only west–east component: $\overline{V} \cdot \overline{\nabla} T = u\dfrac{\partial T}{\partial x}, \quad \dfrac{\partial T}{\partial x} = -\dfrac{2°\mathrm{C}}{100\,\mathrm{km}}$

But temperature in the layer is decreasing $\implies u\dfrac{\partial T}{\partial x} < 0$

Since we are considering a layer, we use mean values:

$$\text{Temperature advection in the layer} = -\overline{u}\frac{\partial \overline{T}}{\partial x}$$

where bars denote the means.

$$\overline{u} = \frac{(u_g(500) + u_g(750))}{2} = \frac{(-14.14 - 14.14)}{2} = -14.14\,\mathrm{m\,s^{-1}}$$

$$\begin{aligned}
\therefore \overline{u}\frac{\partial \overline{T}}{\partial x} &= (-14.14)\left(-\frac{2}{100\,000}\right) \mathrm{m\,s^{-1}\,K\,m^{-1}} \\
&= 2.828 \times 10^{-4}\,\mathrm{K\,s^{-1}} \qquad (\times 3600) \\
&= 1.018\,\mathrm{K\,h^{-1}} \qquad (^{\circ}\mathrm{C\,h^{-1}})
\end{aligned}$$

Therefore, temperature advection in the layer $= -\overline{u}\dfrac{\partial \overline{T}}{\partial x} = -1.018\,\mathrm{K\,h^{-1}}$

Bonus Homework: Describe the relationship between turning of geostrophic wind and temperature advection in terms of backing and veering of the wind with height for the Southern Hemisphere.

Barotropic and baroclinic atmospheres

Barotropic atmosphere: $\rho = \rho(p)$; thermal wind equation $\dfrac{\partial \overline{V}_g}{\partial \ln p} = 0$, which states that the geostrophic wind is independent of height.

Baroclinic atmosphere: $\rho = \rho(p, T)$; geostrophic wind has vertical sheer, related to the horizontal temperature gradient.

Vertical motion

In general the vertical velocity component of synoptic-scale motions is not measured directly, but must be inferred from fields that are measured directly.

Two commonly used methods for inferring vertical motion:

1. Kinematic (based on continuity equation)
2. Adiabatic (based on thermodynamic energy equation)

$$\begin{aligned}
\omega &= \omega(p) \ \text{ vertical velocity in isobaric coordinates.} \\
\omega &\equiv \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \\
&= \frac{\partial p}{\partial t} + \overline{V}\cdot\overline{\nabla}p + w\frac{\partial p}{\partial z} \\
&= \frac{\partial p}{\partial t} + (\overline{V}_g + \overline{V}_a)\cdot\overline{\nabla}p - g\rho w \qquad \left(\text{Since } \frac{\partial p}{\partial z} = -\rho g\right)
\end{aligned}$$

$\overline{V}_a$: Ageostrophic wind, $|\overline{V}_a| \ll |\overline{V}_g|$ the geostrophic wind

$$\therefore \omega = \frac{\partial p}{\partial t} + \overline{V}_g\cdot\overline{\nabla}p + \overline{V}_a\cdot\overline{\nabla}p - g\rho w$$

<u>Bonus Homework</u>: Show that $\overline{V}_g\cdot\overline{\nabla}p = 0$ $\quad \left(\overline{V}_g = \frac{1}{\rho f}\overline{k}\times\overline{\nabla}p\right)$

$$\therefore \omega = \frac{\partial p}{\partial t} + \overline{V}_a\cdot\overline{\nabla}p - g\rho w \tag{3.37}$$

Scale analysis:

$$\begin{aligned}
\frac{\partial p}{\partial t} &\sim 10\,\text{hPa}\,/\,\text{day} \qquad [1\,\text{hPa} = 100\,\text{Pa}] \\
\overline{V}_a\cdot\overline{\nabla}p &\sim (1\,\text{m}\cdot\text{s}^{-1})(0.01\,\text{hPa}\,/\,\text{km}) \sim 1\,\text{hPa}\,/\,\text{day} \\
g\rho w &\sim 100\,\text{hPa}\,/\,\text{day}
\end{aligned}$$

Therefore, a good approximation is

$$\omega = -g\rho w \tag{3.38}$$

Kinematic method

One method of deducing the vertical velocity. Integration of the continuity equation $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$ with respect to pressure from a reference level p_s to any level p, yields

$$w(z) = \frac{\rho(z_s)w(z_s)}{\rho(z)} - \frac{p_s - p}{\rho(z)g}\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right), \tag{3.40}$$

where z and z_s are the heights of pressure levels p and p_s, respectively.

Derivation of (3.40):

(3.5): $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0 \implies \frac{\partial \omega}{\partial p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

Integrate this expression with respect to pressure from a reference level p_s to any level p:

$$\int_{p_s}^{p} \frac{\partial \omega}{\partial p} dp = -\int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dp$$
$$\therefore \omega(p) - \omega(p_s) = -\left[\frac{\partial}{\partial x}\int_{p_s}^{p} u dp + \frac{\partial}{\partial y}\int_{p_s}^{p} v dp\right]$$

Define a pressure weighted vertical average:

$$\langle A \rangle \equiv (p - p_s)^{-1}\int_{p_s}^{p} A dp$$
$$\therefore \int_{p_s}^{p} A dp = (p - p_s)\langle A \rangle = -(p_s - p)\langle A \rangle$$

$$\therefore \omega(p) - \omega(p_s) = -\left[\frac{\partial}{\partial x}\{-(p_s - p)\langle u \rangle\} + \frac{\partial}{\partial y}\{-(p_s - p)\langle v \rangle\}\right]$$
$$= (p_s - p)\frac{\partial \langle u \rangle}{\partial x} + (p_s - p)\frac{\partial \langle v \rangle}{\partial y}$$
$$\therefore \omega(p) = \omega(p_s) + (p_s - p)\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)$$

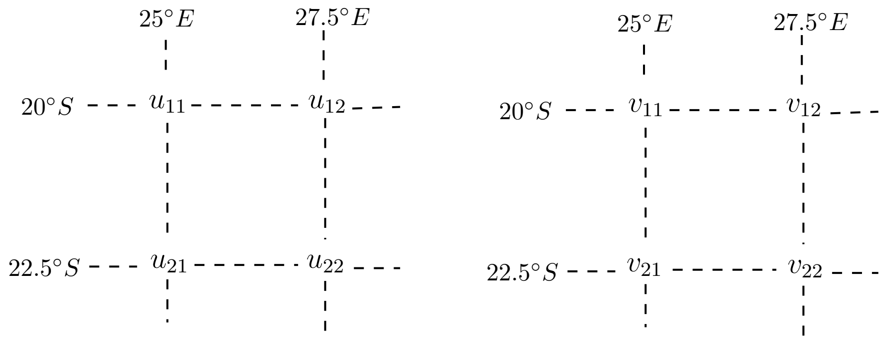
Since $\omega = -\rho g w$, we get $-\rho(z)gw(z) = -\rho(z_s)gw(z_s) + (p_s - p)\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)$

$$w(z) = \frac{\rho(z_s)w(z_s)}{\rho(z)} - \frac{p_s - p}{\rho(z)g}\left(\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y}\right)$$

To infer the vertical velocity from the equation above requires knowledge of the horizontal divergence:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Consider the table below showing the u and v components of the wind $(\mathrm{m \cdot s^{-1}})$ for the $200\,\mathrm{hPa}$ level, on a 2.5° lat–long grid:



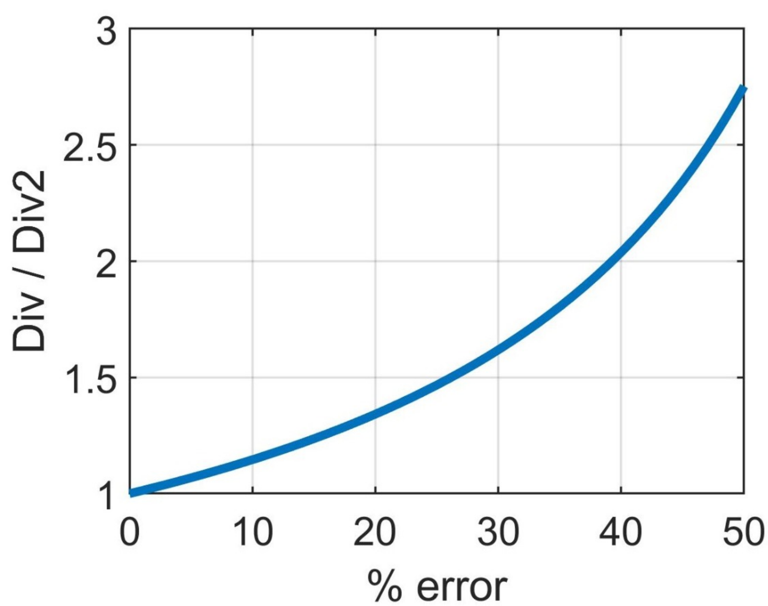
Using finite difference approximations:

$$\frac{\delta u}{\delta x} = [(u_{12} + u_{22})/2 - (u_{11} + u_{21})/2]/\delta x$$
$$\frac{\delta v}{\delta y} = [(v_{11} + v_{12})/2 - (v_{21} + v_{22})/2]/\delta y$$
$$\delta y = 2\pi 6.37 \times 10^6/144; \quad \delta x = \delta y \cos(21.25°)$$

Typical values for u and v:

$$u = [28.8 \quad 28.7; \quad 34.6 \quad 37.4]$$
$$v = [-20.5 \quad -18.2; \quad -26.9 \quad -25.0]$$

An error in one of the wind components can lead to an exponential growth in the estimated divergence. See figure below. For this reason, the continuity equation method is not recommended for estimating the vertical motion field from observed horizontal winds.



Adiabatic method

The **adiabatic method** for inferring vertical velocities, which is not so sensitive to **errors** in the measured horizontal velocities, is based on the thermodynamic energy equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \tag{3.6}$$

If J, the diabatic heating, is small:

$$\begin{aligned}\omega &= \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)\\ &= \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + \overline{V}\cdot\overline{\nabla}T\right)\end{aligned}$$

The temperature advection $\overline{V}\cdot\overline{\nabla}T$ can be accurately obtained from geostrophic winds, and so this method can be applied.

However, $\dfrac{\partial T}{\partial t}$ is difficult to estimate accurately since observations are not typically at close time intervals.

This method is also inaccurate when J is not small (i.e., strong diabatic heating) as is the case of storms in which heavy rainfall occurs over a large area.

Exercise: For a high altitude station located near the 750 to $500\,\mathrm{hPa}$ layer, the temperature is decreasing at a rate of 2°C per hour. Compute the vertical velocity in $\mathrm{cm/s}$ using the adiabatic method. Suppose the lapse rate at the station is 4°C$/\,\mathrm{km}$, temperature advection is $-2.828\times10^{-4}\,\mathrm{K\,s^{-1}}$, and that the dry adiabatic lapse rate is determined by gravity and by the specific heat of dry air at constant pressure.

Solution: $\dfrac{\partial T}{\partial t} = -2°\mathrm{C\,h^{-1}}$ (decreasing)

$$\text{Adiabatic method: } \omega = \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + \overline{V}\cdot\overline{\nabla}T\right), \quad S_p = \frac{\Gamma_d - \Gamma}{\rho g} \text{ and } \omega = -\rho g w$$

$$\therefore w = \frac{\left(\dfrac{\partial T}{\partial t} + \overline{V}\cdot\overline{\nabla}T\right)}{\Gamma - \Gamma_d}$$

$$\begin{aligned}\Gamma_d = \frac{g}{c_p} = \frac{9.81\,\mathrm{m\,s^{-2}}}{1005\,\mathrm{J\,K^{-1}\,kg^{-1}}} &= 9.771\times10^{-3}\,\mathrm{m\,s^{-2}}(\mathrm{J\,K^{-1}\,kg^{-1}})^{-1}\\ &= 9.771\times10^{-3}\,\mathrm{m\,s^{-2}}(\mathrm{kg\,m\,s^{-2}\,m\,K^{-1}\,kg^{-1}})^{-1}\\ &= 9.771\times10^{-3}\,\mathrm{K\,m^{-1}}\end{aligned}$$

$$\Gamma = 4\,\mathrm{K\,m^{-1}} = 4\times10^{-3}\,\mathrm{K\,m^{-1}}$$

$$
\begin{aligned}
w &= \frac{\left(\left(-\dfrac{2}{3600}\right)\mathrm{K\,s^{-1}} - 2.828\times10^{-4}\,\mathrm{K\,s^{-1}}\right)}{(4\times10^{-3} - 9.771\times10^{-3})\,\mathrm{K\,m^{-1}}}\\
&= 0.1453\,\mathrm{m\,s^{-1}}\\
&= 14.53\,\mathrm{cm\,s^{-1}}
\end{aligned}
$$

Surface pressure tendency

The development of a **negative surface pressure tendency** is a classic warning of an **approaching cyclonic weather disturbance**.

$$\omega(p) = \omega(p_s) - \int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp \tag{3.39}$$

$$\text{where } \lim_{p \to 0} \implies 0 = \omega(p_s) + \int_{0}^{p_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp$$

$$\therefore \omega(p_s) = - \int_{0}^{p_s} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp$$

$$= - \int_{0}^{p_s} (\overline{\nabla} \cdot \overline{V}) dp \tag{3.43}$$

$$\omega = \frac{\partial p}{\partial t} + \overline{V}_a \cdot \overline{\nabla} p - g \rho w \tag{3.37}$$

Assumption: w at the surface $= 0$, and $\overline{V}_a \cdot \overline{\nabla} p$ can be neglected (scaling considerations)

$$\therefore \omega \approx \frac{\partial p}{\partial t}$$

$$\therefore \frac{\partial p}{\partial t} \approx - \int_{0}^{p_s} (\overline{\nabla} \cdot \overline{V}) dp \tag{3.44}$$

In words: The surface **pressure tendency** at a given point is determined by the total convergence (negative divergence) of mass into the vertical column of atmosphere above that point.

The utility of the tendency equation is severely limited due to the fact that $\overline{\nabla} \cdot \overline{V}$ is difficult to compute from observations because it depends on the ageostrophic wind field.

Bonus Homework: Describe qualitatively the origin of surface pressure changes and the relationship of such changes to the horizontal divergence.

The circulation theorem

Circulation about a closed contour in a fluid:

$$
\begin{aligned}
C &\equiv \oint \overline{U} \cdot d\bar{l} \qquad (d\bar{l} \text{ is the displacement vector locally tangent to the contour}) \\
&= 2\Omega\pi R^2 \qquad R\text{: radius of circular ring of fluid}
\end{aligned}
$$

$\Longrightarrow$ the circulation is 2π times the angular momentum of the fluid.

By integrating Newton's second law, we can obtain the circulation theorem in an absolute coordinate system as:

$$\frac{DC_a}{Dt} = \frac{D}{Dt}\oint \overline{U}_a \cdot d\bar{l} = -\oint \frac{1}{\rho}dp$$

The solenoidal term is $-\oint \frac{1}{\rho}dp$

In meteorological analysis it is more convenient to work with the relative circulations C.

$$
\begin{aligned}
C &= C_a - C_e \qquad [C_e \text{ : due to Earth's rotation}] \\
&= C_a - 2\Omega A_e \qquad [A \text{ : area}] \\
\frac{DC}{Dt} &= \frac{DC_a}{Dt} - 2\Omega\frac{DA_e}{Dt} \\
&= -\oint \frac{1}{\rho}dp - 2\Omega\frac{DA_e}{Dt}
\end{aligned}
$$

Vorticity

Definition: The microscopic measure of rotation in a fluid. It is a vector field defined as the curl of velocity.

$$\text{Absolute vorticity} \quad \omega_a \equiv \overline{\nabla} \times \overline{U}_a$$

$$\text{Relative vorticity} \quad \omega \equiv \overline{\nabla} \times \overline{U} \qquad (\overline{U} \text{ is the relative velocity})$$

$$\begin{aligned}
\therefore \omega &= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (\bar{i}u + \bar{j}v + \bar{k}w) \\
&= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \bar{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \bar{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \bar{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

For large-scale dynamic meteorology, the concern is only with the vertical components of absolute and relative vorticity.

$$\text{Absolute vorticity} \quad \eta \equiv \overline{k} \cdot (\overline{\nabla} \times \overline{U}_a)$$

$$\text{Relative vorticity} \quad \zeta \equiv \overline{k} \cdot (\overline{\nabla} \times \overline{U})$$

Regions of $\zeta < 0$ are associated with cyclonic storms in the Southern Hemisphere.

The distribution of ζ is an excellent diagnostic for weather analysis.

Planetary vorticity: the local vertical component of the vorticity of the earth due to its rotation

$$\overline{k} \cdot \nabla \times \overline{U}_e = 2\Omega \sin\phi = f, \text{ the Coriolis parameter}$$

$$\eta = \zeta + f$$

$$\begin{aligned}
\zeta &= \overline{k} \cdot \left(\bar{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \bar{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \overline{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) \\
&= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
\therefore \eta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f
\end{aligned}$$

Exercise 1: What is the relative vorticity on the side of a current which decreases in magnitude towards the south at a rate of $10\,\mathrm{m}\,/\,\mathrm{s}$ for every $500\,\mathrm{km}$?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

No west–east component: $\dfrac{\partial v}{\partial x} = 0$

$$\begin{aligned}
&\frac{\partial u}{\partial y} < 0 \qquad \text{(towards the south)} \\
\therefore\ &\frac{\partial u}{\partial y} = -\frac{10\,\mathrm{m\,s^{-1}}}{500\,000\,\mathrm{m}} = -2 \times 10^{-5}\,\mathrm{s^{-1}} \\
\therefore\ &\zeta = 0 - (-2 \times 10^{-5}\,\mathrm{s^{-1}}) \\
&\ \ = 2 \times 10^{-5}\,\mathrm{s^{-1}}
\end{aligned}$$

Exercise 2: An air parcel at 30°S moves southward conserving absolute vorticity (the initial absolute vorticity is equal to the final absolute vorticity). If its initial relative vorticity is $5 \times 10^{-5}\,\mathrm{s^{-1}}$, what is its relative vorticity upon reaching 90°S?

Solution:

$$(\zeta + f)_{\text{initial}} = (\zeta + f)_{\text{final}}$$

$$\begin{aligned}
f_{\text{initial}} &= 2\Omega \sin(-30°) = 2\Omega \left(-\frac{1}{2}\right) = -\Omega \\
f_{\text{final}} &= 2\Omega \sin(-90°) = 2\Omega\,(-1) = -2\Omega \\
\zeta_{\text{initial}} &= 5 \times 10^{-5}\,\mathrm{s^{-1}} \quad \text{(given)}
\end{aligned}$$

$$\begin{aligned}
\zeta_{\text{final}} &= (\zeta + f)_{\text{initial}} - f_{\text{final}} \\
&= 5 \times 10^{-5} - \Omega - (-2\Omega) \\
&= 5 \times 10^{-5} + \Omega \\
&= 5 \times 10^{-5} + 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}} \\
&= 12.292 \times 10^{-5}\,\mathrm{s^{-1}}
\end{aligned}$$

Bonus Homework: Determine the relationship between relative vorticity and relative circulation (macroscopic).

Potential vorticity

Definition and characteristics of potential vorticity

The potential vorticity (PV) is the absolute circulation of an air parcel that is enclosed between two isentropic surfaces (a surface in space on which potential temperature is everywhere equal). If PV is displayed on a surface of constant potential temperature, then it is officially called IPV (isentropic potential vorticity). PV could also be displayed on another surface, for example a pressure surface. Note from the relation below, that PV is simply the product of absolute vorticity on an isentropic surface and static stability. So PV consists, in contrast to vorticity on isobaric surfaces, of two factors, a dynamical element and a thermodynamical element.

$$PV \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right)$$

where,

f is the Coriolis parameter

g is the gravitational acceleration

p is the pressure

PV is the potential vorticity

θ is the potential temperature: $\theta = T\left(\frac{p_s}{p}\right)^{R/c_p}$

ζ_θ is the relative isentropic vorticity [the vertical component of relative vorticity evaluated on an isentropic surface]

Within the troposphere, the values of PV are usually low. However, the potential vorticity increases rapidly from the troposphere to the stratosphere due to the significant change of the static stability. Typical changes of the potential vorticity within the area of the tropopause are from 1 (tropospheric air) to 4 (stratospheric air) PV units (PV unit: $1\,\mathrm{PVU} = 10^{-6}\,\mathrm{K\,kg^{-1}\,m^2\,s^{-1}}$). Today in most of the literature the 2 PV unit anomaly, which separates tropospheric from stratospheric air, is referred to as dynamical tropopause. The traditional way of describing the tropopause, is with use of the potential temperature or static stability. This is only a thermodynamical way of characterising the tropopause. The benefit of using PV is that the tropopause can be understood in both thermodynamic and dynamic terms. An abrupt folding or lowering of the dynamical tropopause can also be called an upper PV-anomaly. When this occurs, stratospheric air penetrates into the troposphere resulting in high values of PV with respect to the surroundings, creating a positive PV-anomaly.

In the lower levels of the troposphere, strong baroclinic zones often occur which can be regarded as low level PV anomalies.

It must be stressed that this other way of looking at the dynamics of the atmosphere will not necessarily result in new conclusions. However, it may give new dimensions to things that, in fact, were already known.

The two main advantages of potential vorticity (with certain assumptions) are: conservation and invertibility. The two advantages will be discussed briefly:

Conservation

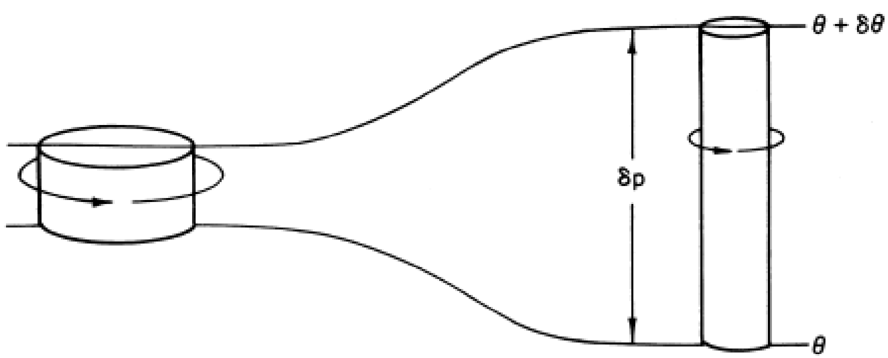
With the following assumptions PV is a conserved parameter:

1. Adiabatic stream (no diabatic heating or cooling)
2. No friction
3. Homogenous
4. Non-compressing

A first mathematical consequence of the conservation can be derived from the definition of PV: A parcel will keep the same value of PV if it moves along an adiabat through the atmosphere thus the equation for PV can be written as:

$$PV \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{constant} \tag{4.12}$$

Due to the conservation of PV, there is a close relationship between absolute vorticity and static stability (the ability of a fluid at rest to become turbulent or laminar [flow taking place along constant streamlines, without turbulence] due to the effects of buoyancy). The diagram below shows a parcel (cylinder) that is confined between potential temperature (isentropic) surfaces θ and $\theta + \delta\theta$ which are separated by a pressure interval δp. Difference in potential temperature between the top and bottom is the same for the two cylinders. If PV is conserved, and the cylinder is stretched, then static stability is decreasing and absolute vorticity must increase. Alternatively, if one goes from the stretched cylinder to the squashed cylinder, then static stability is increasing and absolute vorticity must decrease.



Due to the conservation of PV, significant features that are related to synoptic scale weather systems can be identified and followed in space as well as in time. This is a very powerful characteristic of this property.

Especially the case of a lowering of the dynamical tropopause, the upper PV-anomaly can be followed in time and space rather easily. PV anomalies are well related to a lot of dynamical processes in the troposphere. A distinct example of this are cases of Rapid Cyclogenesis where PV-anomalies play an important role.

The sudden creation or destruction of PV means that diabatic processes are involved (release of latent heat, friction, radiation). This fact can be used as tool to identify or even quantify the influence of these processes.

Invertibility

The second advantage of PV, invertibility, is a very important tool, because it allows one to obtain familiar meteorological fields, like the geopotential, wind, temperature and the static stability, when the distribution of the PV and the boundary conditions, potential temperature at the surface, are known. Further with the help of the invertibility it is possible to quantify the importance of PV-anomalies and the strength of their associated circulation and/or temperature pattern.

Inverting the PV for the entire atmosphere is interesting, but a more insightful diagnostic technique is piecewise PV inversion (PPVI). This involves dividing the atmosphere into significant layers and independently inverting the PV in those layers. This technique allows for analysis of the influence of discreet portions of the total PV field on the flow throughout the domain.

PV-thinking in the real atmosphere

The dynamical tropopause

The tropopause separates the well-mixed troposphere with the highly stratified, statically stable stratosphere. The tropopause is conventionally thought of from a thermal point of view and is based on the vertical temperature lapse rate. However, since high-PV values are generally associated with highly statically stable air, the tropopause can also be defined by the isentropic (contours of constant potential temperature) gradient of PV. The PV definition of the troposphere is known as the dynamical tropopause. By convention, the dynamical tropopause is usually defined by a constant PV contour which separates tightly packed PV contours of the stratosphere and low vertical gradient PV contours of the troposphere. A value between -1.5 and -2.5 PVU is most commonly used.

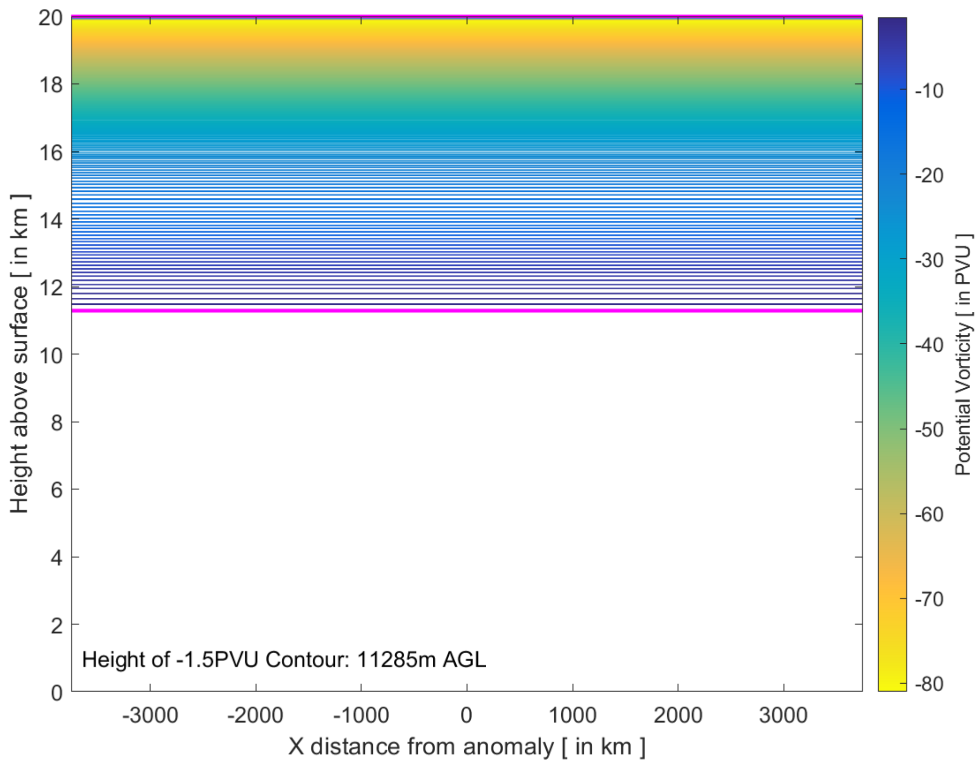


Figure 7: Cross-sectional PV in an idealised, mean atmosphere.

PV anomalies

Mathematically, an anomaly is the departure of a value from the mean distribution. A high-PV anomaly will thus be where there are anomalously high values (large negative values in the southern hemisphere) of PV compared to the mean distribution. Conversely, a low-PV anomaly will have anomalously small values (smaller negative values) of PV compared to the mean distribution.

Upper level PV anomalies

As seen in Fig. 7, there exists a reservoir of high-PV air in the stratosphere. Thus, stratospheric air is a source of high-PV anomalies in the troposphere. Upper-level high-PV anomalies can therefore be viewed, from a cross-sectional point of view, as tongues of high-PV stratospheric air intruding into the troposphere towards the surface. An idealised example of this is shown in Fig. 8. We recall that the circulation can be inferred from the PV distribution by the power of PV inversion and recall that PV can be represented by equation (4.12). The high-PV (negative PV anomaly) induces a negative vorticity anomaly. Flow is cyclonic around a negative vorticity anomaly in the Southern Hemisphere and hence cyclonic around the high-PV anomaly. Since the atmosphere is in thermal wind balance, the velocity of the circulation above and below the anomaly will also be cyclonic but the flow will be weaker.

The fact that the atmosphere is in thermal wind balance allows for us to decipher the temperature structure of the sectors. Recalling that the definition of the thermal wind is the difference between the upper and lower wind vectors ($\overline{V}_T = \overline{V}_g(p_1) - \overline{V}_g(p_0)$) and that the cold pool lies to the left (right) of the thermal

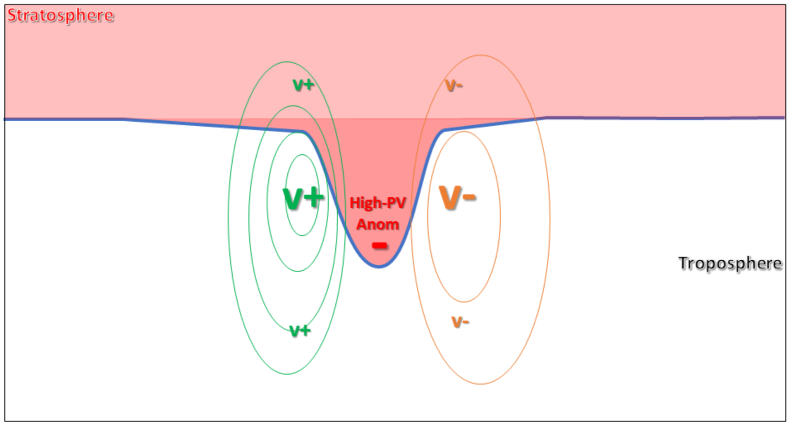


Figure 8: A cross-sectional view of an idealised PV intrusion in the upper troposphere inducing upper-level cyclonic flow around it.

wind vector in the Northern (Southern) hemisphere, it follows that there must exist a cold pool below the high-PV anomaly. Similarly, there must exist a warm pool in the stratospheric sector above the high-PV anomaly with cold air surrounding it. Thus, the potential temperature structure will look as below.

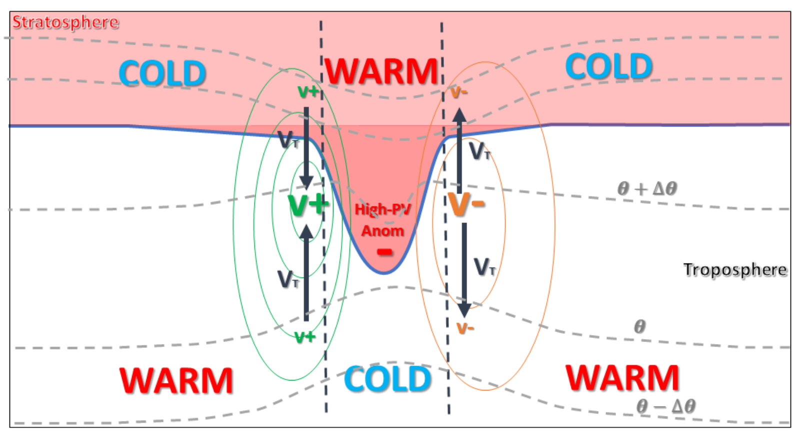


Figure 9: Cross-sectional view of the thermal structure of an idealised PV intrusion in the upper troposphere inducing upper-level cyclonic flow around it.

Low-level and surface PV anomalies

PV anomalies are not confined to the upper troposphere. High-PV structures can also be found in the low levels, often associated with diabatic processes. The PV anomalies act in a similar way to their upper-level counterparts, stimulating cyclonic flow around them. Similar arguments with respect to the thermal balance of the atmosphere can be made in order to understand the thermal structure surrounding the anomaly as well as the cyclonic flow that is induced throughout the atmosphere.

You will recall that for PV to be conserved, the flow must be both frictionless and adiabatic. At the surface, this is not strictly true. Thus, PV cannot be directly used on the surface and we need to use a PV-like parameter to analyse the surface is in a PV-thinking framework. It has been shown that surface potential temperature (θ) anomalies can act as PV-like anomalies. Warm θ anomalies behave in a similar way to high-PV anomalies on the surface where cyclonic flow is stimulated around a warm θ anomaly and anti-cyclonic flow results from cold θ anomalies.

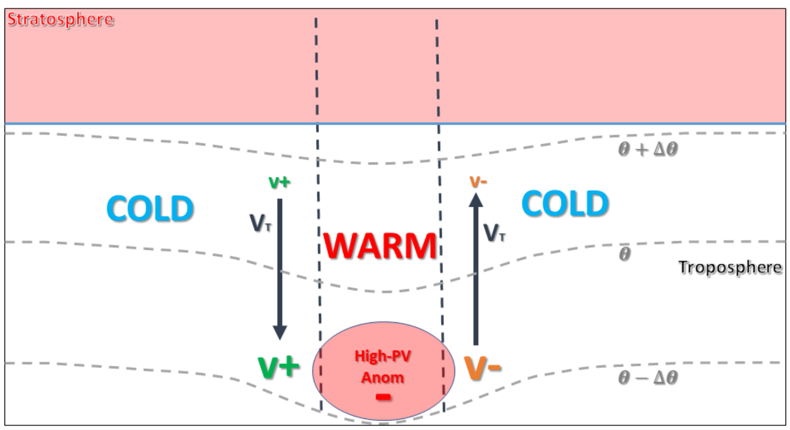


Figure 10: A cross-sectional view of an idealised PV intrusion in the lower troposphere inducing low-level cyclonic flow around it.

Interaction of anomalies

As can be shown schematically in Figure 8 and Figure 9, cyclonic circulation around the upper-level intrusion of high-PV stratospheric air into the upper troposphere is not confined to the upper troposphere. Mirrored, although weaker, cyclogenetic forcing is also present on the surface. As a result of the surface temperature gradient, the low-level cyclogenetic forcing results in warm air temperature advection ahead of the upper-level PV intrusion axis. This results in a warm potential temperature anomaly ahead of the upper-level PV axis. Recall that warm potential temperature anomalies on the surface can be interpreted to be similar to high-PV anomalies whereby they can induce cyclonic circulation around them. The cyclogenetic forcing is induced throughout the troposphere above the anomaly, with mirrored cyclogenetic forcing stimulated ahead of the upper level PV intrusion in the upper-levels. Whilst the surface anomaly lies ahead of the upper-level anomaly there is positive feedback between the two anomalies and thus are mutually beneficial to one another. Low-level anomalies induced by diabatic processes can further add to the development of the surface cyclone. The phase-locked alignment of all 3 of these anomalies is known as a "PV tower" and can lead to explosive cyclogenesis.

In the atmosphere, these processes lead to the development of baroclinic weather systems such as mid-latitude cyclones or cut-off lows that extend to the surface, where the system leans westward with height.

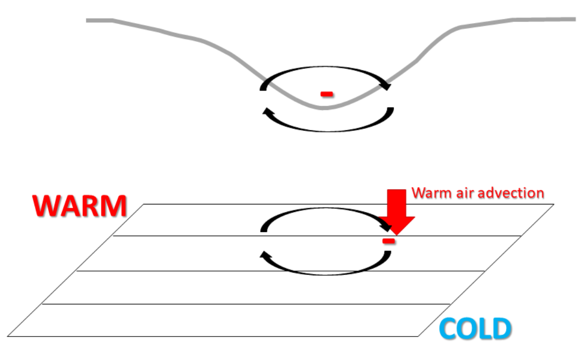


Figure 11: Adapted from Hoskins *et al.* (1985). Interaction between an upper air intrusion of high-PV air and an induced surface high-PV anomaly.

PV on isentropic maps

Isentropic surfaces, lines of constant potential temperature, are frequently used in the dynamical meteorological analyses. The analysis of isentropic PV has many applications including the identification of Rossby wave breaking (RWB) in the upper troposphere. Upper-level PV intrusions are easily identifiable on isentropic surfaces. Potential temperature contours slant surface-ward from the poles to the equator. Thus, an isentropic contour will cut through the quasi-horizontal dynamical tropopause at some point between the pole and the equator. High-PV values (stratospheric air) will be found towards the poles whilst low-PV values (tropospheric air) will be found towards the equator. A PV anomaly in the upper troposphere can be seen in the isentropic PV field as a tongue of high-PV, stratospheric air extending towards the equator.

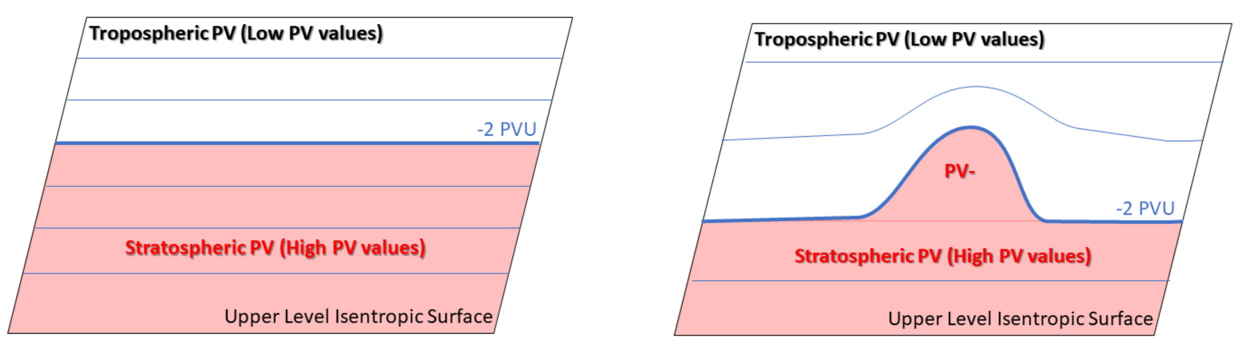


Figure 12: Left: Climatological mean isentropic surface of the upper troposphere. Right: An idealised PV intrusion in the upper troposphere as seen on upper-level isentropic PV surface.

Useful additional reading:

Lackmann (2011) - Midlatitude synoptic meteorology: Dynamics, analysis and forecasting (Chapter 4)

Hoskins *et al.* (1985) - On the use and significance of isentropic potential vorticity maps

Barnes *et al.* (2021) - Cape storm: A dynamical study of a cut-off low and its impact on South Africa

For more information on PV, follow this link: `http://www.zamg.ac.at/docu/Manual/SatManu/main.htm?/docu/Manual/SatManu/Basic/Parameters/PV.htm`

For real world examples related to the material above, follow this link: `https://weathermanbarnes.github.io/UPDynamicalForecasts`

The vorticity equation

Objective: Derive an equation for the time rate of change of vorticity without limiting the validity to adiabatic motion.

Cartesian coordinate form

Approximate horizontal momentum equations:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{[zonal component equation]}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{[meridional component equation]}$$

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) = \frac{\partial}{\partial y}(fv) - \frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = -\frac{\partial}{\partial x}(fu) - \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial y}(fv) = -\frac{\partial}{\partial y}\left(\rho^{-1}\frac{\partial p}{\partial x}\right)$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial x}(fu) = -\frac{\partial}{\partial x}\left(\rho^{-1}\frac{\partial p}{\partial y}\right)$$

$$\therefore \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y \partial z} - v\frac{\partial f}{\partial y} - f\frac{\partial v}{\partial y}$$
$$= -\frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} \qquad (1)$$

and

$$\therefore \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial x \partial z} + u\overbrace{\frac{\partial f}{\partial x}}^{=0} + f\frac{\partial u}{\partial x}$$
$$= -\frac{\partial \rho^{-1}}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} \qquad (2)$$

(2) − (1): LHS

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right]+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+v\frac{\partial f}{\partial y}$$

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial x}\qquad y=\rho^{-1}\quad u=\rho$$
$$\frac{\partial}{\partial x}\rho^{-1}=\frac{\partial}{\partial \rho}\rho^{-1}\frac{\partial \rho}{\partial x}$$
$$=-\rho^{-2}\frac{\partial \rho}{\partial x}$$

(2) − (1): RHS

$$-\frac{\partial \rho^{-1}}{\partial x}\frac{\partial p}{\partial y}-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{\partial \rho^{-1}}{\partial y}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}=-(-1)\rho^{-2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}+(-1)\rho^{-2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$
$$=\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$

Consider

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Since $\zeta=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$
$$+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Since $f=f(y)$, $\frac{Df}{Dt}=0+0+v\frac{\partial f}{\partial y}+0$

$$\therefore\ \frac{D\zeta}{Dt}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{Df}{Dt}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\implies \frac{D}{Dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)\qquad(4.17)$$

The rate of change of absolute vorticity following the motion is given by the sum of the divergence, the tilting or twisting, and the solenoidal terms.

Scale analysis of the vorticity equation

Characteristic scales for the field variables based on typical observed magnitudes for **synoptic-scale** motions:

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical scale
$L \sim 10^6\,\mathrm{m}$	length scale
$H \sim 10^4\,\mathrm{m}$	depth scale
$\delta p \sim 10\,\mathrm{hPa}$	horizontal pressure scale
$\rho \sim 1\,\mathrm{kg\,m^{-3}}$	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
$L/U \sim 10^5\,\mathrm{s}$	time scale
$f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$	Coriolis parameter
$\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$	"beta" parameter

Use an advective time scale because the vorticity pattern tends to move at a speed comparable to the horizontal wind speed.

First, the relative vorticity equation $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \lesssim \frac{U}{L} \sim (10^5\,\mathrm{s})^{-1} = 10^{-5}\,\mathrm{s}^{-1}$

[$\lesssim$ means less than or equal to in order of magnitude]

The magnitude of the terms of the equation below will be evaluated:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

The Rossby number $\mathrm{Ro} \equiv \frac{U}{f_0 L}$

and since $\zeta \lesssim \frac{U}{L}$, $\frac{\zeta}{f_0} \lesssim \frac{U}{f_0 L} \sim 10^{-1}$ $\left[10\,\mathrm{m\cdot s^{-1}}/(10^4\,\mathrm{s^{-1}}\,10^6\,\mathrm{m})\right]$

$$\therefore \zeta \sim \frac{1}{10} f_0$$

$$\therefore (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Note: Small Ro signifies a system which is strongly affected by Coriolis forces, and a large Ro a system in which inertial and centrifugal forces dominate. In tornadoes $\mathrm{Ro} \approx 10^3$; in low pressure systems $\mathrm{Ro} \approx$ 0.1–1.

Near the centre of intense cyclonic storm $\left|\frac{\zeta}{f}\right| \sim 1$, the relative vorticity should be retained.

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} &\sim \frac{U/L}{L/U} = \frac{U^2}{L^2} \sim 10^{-10}\,\mathrm{s^{-2}} \qquad \left[\frac{(10\,\mathrm{m\,s^{-1}})^2}{(10^6\,\mathrm{m})^2} = 10^{-10}\,\mathrm{s^{-2}}\right] \\
u\frac{\partial\zeta}{\partial x} &\sim U\frac{U}{L}\frac{1}{L} = \frac{U^2}{L^2} \\
v\frac{\partial\zeta}{\partial y} &\sim U\frac{U}{L}\frac{1}{L} = \frac{U^2}{L^2} \\
w\frac{\partial\zeta}{\partial z} &\sim W\frac{U}{L}\frac{1}{H} \sim \frac{10^{-2}\,\mathrm{m\,s^{-1}}\,10\,\mathrm{m\,s^{-1}}}{10^6\,\mathrm{m}\,10^4\,\mathrm{m}} = 10^{-11}\,\mathrm{s^{-2}} \\
v\frac{df}{dy} &\sim U\beta \sim 10\,\mathrm{m\,s^{-1}}\,10^{-11}\,\mathrm{m^{-1}\,s^{-1}} = 10^{-10}\,\mathrm{s^{-2}}
\end{aligned}
$$

$$
\begin{cases}
f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim f_0\left(\frac{U}{L} + \frac{U}{L}\right) \sim f_0\frac{U}{L} \sim 10^{-4}\,\mathrm{s^{-1}}\,10\,\mathrm{m\,s^{-1}}\,10^{-6}\,\mathrm{m^{-1}} = 10^{-9}\,\mathrm{s^{-2}} \\
\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \lesssim \frac{W}{L}\frac{U}{H} \sim \frac{10^{-2}\,\mathrm{m\,s^{-1}}\,10\,\mathrm{m\,s^{-1}}}{10^6\,\mathrm{m}\,10^4\,\mathrm{m}} = 10^{-11}\,\mathrm{s^{-2}} \\
\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right) \lesssim \frac{1}{\rho^2}\left(\frac{\delta\rho}{L}\frac{\delta p}{L}\right) = \frac{\delta\rho}{\rho^2}\frac{\delta p}{L^2} \sim \frac{10^{-2}}{\rho}\frac{10\,\mathrm{hPa}}{10^{12}\,\mathrm{m^2}} \sim \frac{10^{-2}10^3\,\mathrm{Pa}}{1\,\mathrm{kg\,m^{-3}}\,10^{12}\,\mathrm{m^2}}
\end{cases} \tag{*}
$$

Consider

$$
\begin{aligned}
1\,\mathrm{Pa} &= 1\,\mathrm{N\,m^2} \\
&= 1(\mathrm{kg\,m\,s^{-2}})\,\mathrm{m^{-2}}
\end{aligned}
$$

$$
\therefore \frac{\delta\rho}{\rho^2}\frac{\delta p}{L^2} \sim 10^{-11}\frac{\mathrm{kg\,m^{-1}\,s^{-2}}}{\mathrm{kg\,m^{-1}}} = 10^{-11}\,\mathrm{s^{-2}}
$$

(*): The inequality ($\lesssim$) is used here because in each case it is possible that the two parts of the expression **might partially cancel** so that the actual magnitude would be less than indicated.

If $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are not nearly equal and opposite (i.e., divergence > 0) the divergence term would be an order of magnitude greater than the other terms (because $f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim 10^{-9}\,\mathrm{s^{-2}}$, followed by terms $\lesssim 10^{-10}$ and smaller).

Therefore, scale analysis of the vorticity equation indicates that synoptic-scale motions must be quasi-nondivergent. The divergence term will be small enough to be balanced by the vorticity advection terms only if: $\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \lesssim 10^{-6}\,\mathrm{s^{-1}}$ since $f_0 \sim 10^{-4}\,\mathrm{s^{-1}}, f_0\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \sim 10^{-10}\,\mathrm{s^{-2}}$

$\implies$ The horizontal divergence must be small compared to the vorticity in synoptic-scale systems.

Retaining only the terms of order $10^{-10}\,\mathrm{s^{-2}}$ in the vorticity equation:

$$
\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = 0
$$

Remember that $v\dfrac{df}{dy} = \dfrac{Df}{Dt}$ and for horizontal motion $v\dfrac{df}{dy} = \dfrac{D_h f}{Dt}$

$$\therefore \frac{D_h\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{D_h f}{Dt} = 0 \qquad \left[\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]$$

$$\implies \frac{D_h}{Dt}(\zeta + f) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.22a}$$

for synoptic-scale motions.

In intense cyclonic storms $|\zeta/f| \sim 1$:

$$\implies \frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.22b}$$

Equation (4.22a) states that the change of absolute vorticity following the horizontal motion on the synoptic scale is given approximately by the concentration or dilution of planetary vorticity caused by the convergence or divergence of the horizontal flow, respectively. In (4.22b), however, it is the concentration or dilution of absolute vorticity that leads to changes in absolute vorticity following the motion.

The form of the vorticity equation given in (4.22b) also indicates why cyclonic disturbances can be much more intense than anti-cyclones. For a fixed amplitude of convergence, relative vorticity will increase, and the factor $(\zeta + f)$ becomes larger, which leads to even higher rates of increase in the relative vorticity. For a fixed rate of divergence, however, relative vorticity will decrease, but when $\zeta \to -f$, the divergence term on the right approaches zero and the relative vorticity cannot become more negative no matter how strong the divergence (This difference in the potential intensity of cyclones and anti-cyclones was discussed in Section 3.2.5 of Holton 4 in connection with the gradient wind approximation).

The approximate forms given in (4.22a) and (4.22b) do not remain valid, however, in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only $\sim 100\,\mathrm{km}$ and the vertical velocity scale is $\sim 10\,\mathrm{cm\,s^{-1}}$. For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

Vorticity in barotropic fluids

The barotropic (Rossby) potential vorticity equation

The velocity divergence form of the continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \overline{\nabla} \cdot \overline{V} = 0 \tag{2.31}$$

For a homogenous incompressible fluid $\frac{D\rho}{Dt} = 0$

$$\therefore \overline{\nabla} \cdot \overline{V} = 0$$
$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$
$$\therefore \frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(-\frac{\partial w}{\partial z}\right)$$

In a barotropic fluid, we let the vorticity be approximated by ζ_g and the wind by (u_g, v_g).

$$\frac{D_h}{Dt}(\zeta_g + f) = (\zeta_g + f)\frac{\partial w}{\partial z}$$

Integrate vertically from z_1 to z_2:

$$\int_{z_1}^{z_2} \frac{D_h}{Dt}(\zeta_g + f)dz = \int_{z_1}^{z_2} (\zeta_g + f)\frac{\partial w}{\partial z}dz$$
$$h\frac{D_h}{Dt}(\zeta_g + f) = (\zeta_g + f)\left[w(z_2) - w(z_1)\right]$$
$$\text{where} \quad h = h(x, y, t); \qquad w \equiv \frac{Dz}{Dt}$$
$$= z_2 - z_1$$

$$\therefore w(z_2) - w(z_1) = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{D_h h}{Dt}$$

Since $h\frac{D_h}{Dt}(\zeta_g + f) = (\zeta_g + f)\left[w(z_2) - w(z_1)\right] = (\zeta_g + f)\frac{D_h h}{Dt}$

$$\therefore \frac{1}{(\zeta_g + f)}\frac{D_h}{Dt}(\zeta_g + f) = \frac{1}{h}\frac{D_h h}{Dt}$$

$$\therefore \frac{D_h}{Dt}(\ln(\zeta_g + f)) - \frac{D_h}{Dt}(\ln h) = 0$$

$$\therefore \frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0 \tag{4.26}$$

which is the potential vorticity conservation theorem for a barotropic fluid.

The quantity conserved following the motion in (4.26) is the Rossby potential vorticity.

Note the following:

$$\frac{D}{Dt}\left(\ln(\zeta_g + f)\right) - \frac{D}{Dt}(\ln h) = 0$$

Therefore,

$$\frac{D}{Dt}(\ln(\zeta_g + f) - \ln h) = 0$$

$$\therefore \frac{D}{Dt}\left(\ln\frac{\zeta_g + f}{h}\right) = 0$$

From calculus: $D_x \ln[f(x)] = \frac{f'(x)}{f(x)}$

$$\implies \frac{D}{Dt}\left(\ln\frac{\zeta_g + f}{h}\right) = \frac{1}{\frac{\zeta_g + f}{h}}\frac{D}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

$$\therefore \frac{D}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

Exercise: By considering the essence of potential vorticity (a measure of the constant ratio of the absolute vorticity to the effective depth of the vortex), an air column at 60°S with initial relative vorticity equal to zero, stretches from sea-level to a fixed tropopause level of $10\,\mathrm{km}$ in height. If the air column moves until it is over a mountain range $2.5\,\mathrm{km}$ high at 45°S, what is its 1) absolute vorticity and 2) relative vorticity as it passes the mountain top?

Solution: $\frac{\zeta + f}{H} = \text{constant} \implies \left(\frac{\zeta + f}{H}\right)_{\text{initial}} = \left(\frac{\zeta + f}{H}\right)_{\text{final}}$, and $\zeta_{\text{initial}} = 0$ (given)

$$f_{\text{initial}} = 2\Omega\sin(-60°) = -1.263 \times 10^{-4}\,\mathrm{s}^{-1}$$
$$f_{\text{final}} = 2\Omega\sin(-45°) = -1.031 \times 10^{-4}\,\mathrm{s}^{-1}$$

$$\begin{aligned}\therefore (\zeta + f)_{\text{final}} &= \frac{H_{\text{final}}}{H_{\text{initial}}}f_{\text{initial}} \\ &= \frac{10 - 2.5}{10}(-1.263 \times 10^{-4}) \\ &= -9.473 \times 10^{-5}\,\mathrm{s}^{-1}\end{aligned}$$

$$\therefore \zeta_{\text{final}} = -9.473 \times 10^{-5}\,\text{s}^{-1} - (-1.031 \times 10^{-4}\,\text{s}^{-1})$$
$$= 8.37 \times 10^{-6}\,\text{s}^{-1}$$

The barotropic vorticity equation

(4.23): $\dfrac{D_h}{Dt}(\zeta + f) = (\zeta + f)\dfrac{\partial w}{\partial z}$

If the flow is purely horizontal, as is the case for barotropic flow in a fluid of constant depth, the divergence term vanishes since $w = 0$.

As before, let vorticity be approximated by ζ_g:

$$\frac{D_h}{Dt}(\zeta_g + f) = 0$$

which states that absolute vorticity is conserved following the horizontal motion.

More generally, absolute vorticity is conserved for any fluid layer in which the divergence of the horizontal wind vanishes, without the requirement that the flow be geostrophic.

For horizontal motion that is nondivergent the flow can be represented by a stream function $\psi(x, y)$ such that $u = -\dfrac{\partial \psi}{\partial y}$ and $v = \dfrac{\partial \psi}{\partial x}$.

$$\begin{aligned} \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) \\ &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \\ &\equiv \nabla^2 \psi \end{aligned}$$

Not a requirement for flow to be geostrophic: $\dfrac{D_h}{Dt}(\zeta + f) = 0$

$$\therefore \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0$$

$$\therefore \frac{\partial}{\partial t}\nabla^2\psi + \overline{V}_\psi \cdot \overline{\nabla}(\nabla^2\psi) + \overline{V}_\psi \cdot \overline{\nabla}(f) = 0; \qquad \overline{V}_\psi = \overline{k} \times \overline{\nabla}\psi$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\overline{V}_\psi \cdot \overline{\nabla}(\nabla^2\psi + f) \tag{4.28}$$

$\implies$ The local tendency of relative vorticity is given by the advection of absolute vorticity.

Because the flow in the mid-troposphere is often nearly nondivergent on the synoptic scale, (4.28) provides a good model for short-term forecasts of the synoptic-scale $500\,\text{hPa}$ flow field.