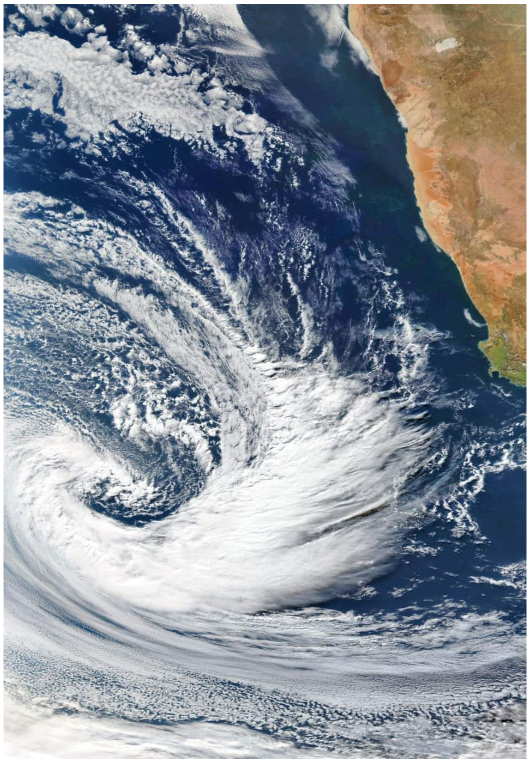
Quasi-Geostrophic Analysis



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Bjerknes-Holmboe theory

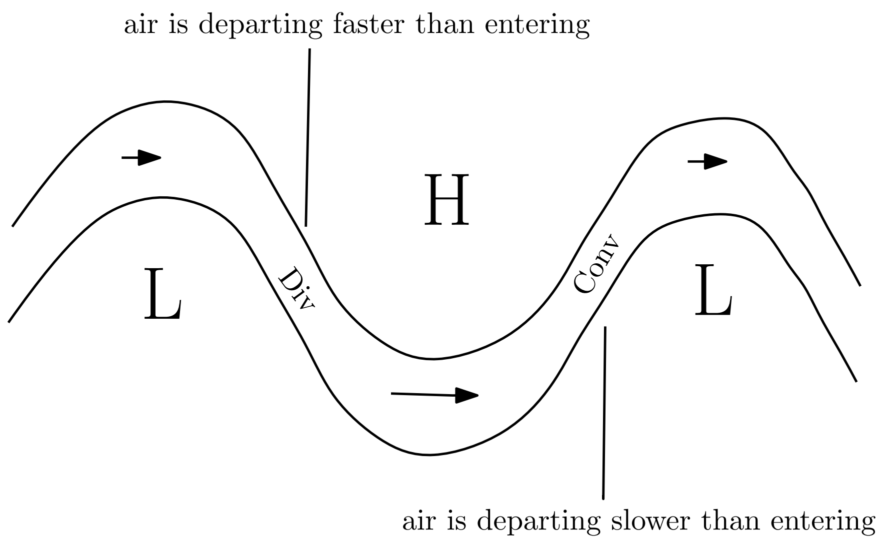
As an introduction to the notion of mid-latitude developing baroclinic systems, we introduce a theory of relating the horizontal distribution of divergence and convergence to a pattern of high and low pressure systems. Highs will move towards regions of convergence (rising pressure), and lows towards regions of divergence (falling pressure). This theory is commonly known as the Bjerknes-Holmboe theory. Here we discuss it only from a qualitative point of view.

Since the divergence of the geostrophic wind (V_g) is zero (for constant f), and the divergence of the gradient wind (V) is not zero, we will examine the pattern of divergence of idealistic pressure fields for gradient flow. Our weather pattern has

1. Sinusoidal $500\mathrm{hPa}$ contours extending from west to east,
2. Circular concentric isobars at the surface.

The curvature effect

We have already shown that $V_g > V$ for cyclonic flow, and $V_g < V$ for anticyclonic flow. Therefore, owing to this curvature effect, we expect a distribution of wind speeds as shown in this figure (the arrows represent the geostrophic wind).



Such a pattern would lead to falling pressure east of the troughs and rising pressure east of the ridge. The expectation is for the pressure system to move eastward because the lows (highs) move towards regions of falling (rising) pressure. Moreover, for a given fixed amplitude of such systems, short wavelengths and high wind speeds, the curvature effect results in an eastward moving wave.

The latitude effect

Assume that all other parameters are kept constant, then the geostrophic and gradient wind speeds decrease with increasing (equatorward) latitude. To demonstrate this statement, consider the gradient wind equation

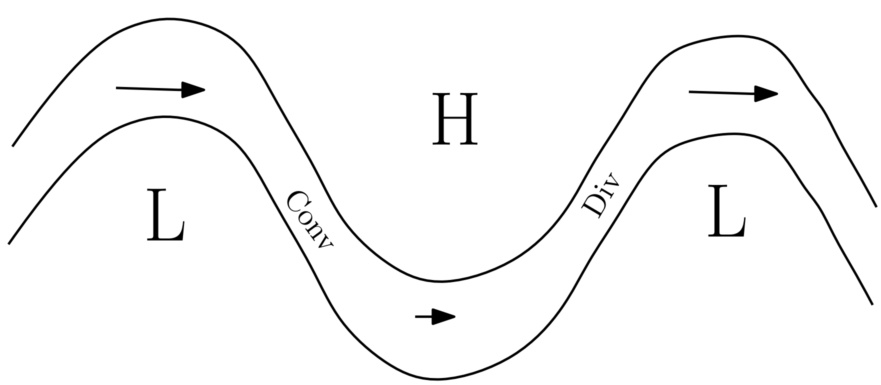
$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$
$$\therefore V = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial n} + \frac{V^2}{R}\right)$$

Apply scale analysis to $\left|\frac{V^2}{R}\right| \simeq \frac{(10\,\mathrm{m\,s^{-1}})^2}{10^6\,\mathrm{m}} = 10^{-4}\,\mathrm{m\,s^{-2}}$, the centrifugal force.

Since a typical parameter value for $\left|\frac{\partial \Phi}{\partial n}\right| = 10^{-3}\,\mathrm{m\,s^{-2}}$, the centrifugal force is about a tenth of the pressure gradient force. This result implies that $V \simeq -\frac{1}{f}\frac{\partial \Phi}{\partial n} = V_g$, which further implies that both the gradient and geostrophic wind increase or decrease similarly for a variable Coriolis parameter. Consider the following table of approximate gradient wind speeds with increasing latitude in the Southern Hemisphere

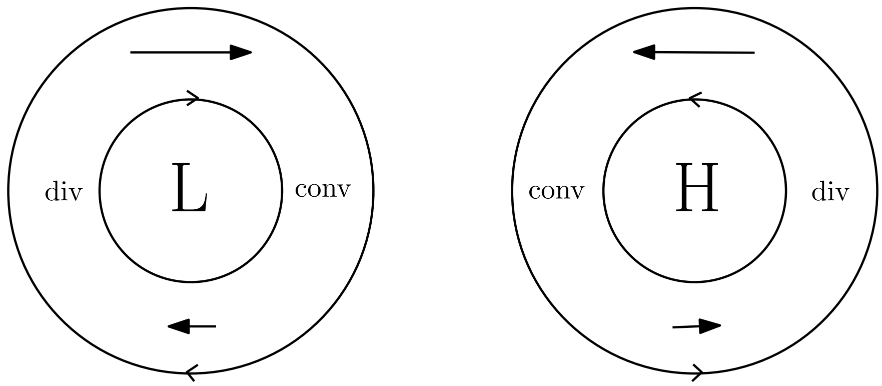
Latitude	Approximate gradient wind speed $(\mathrm{m\,s^{-1}})$
-30	13.7
-45	9.7
-60	7.9

From the gradient wind relationship and the table above, wind speed decreases with increasing latitude.



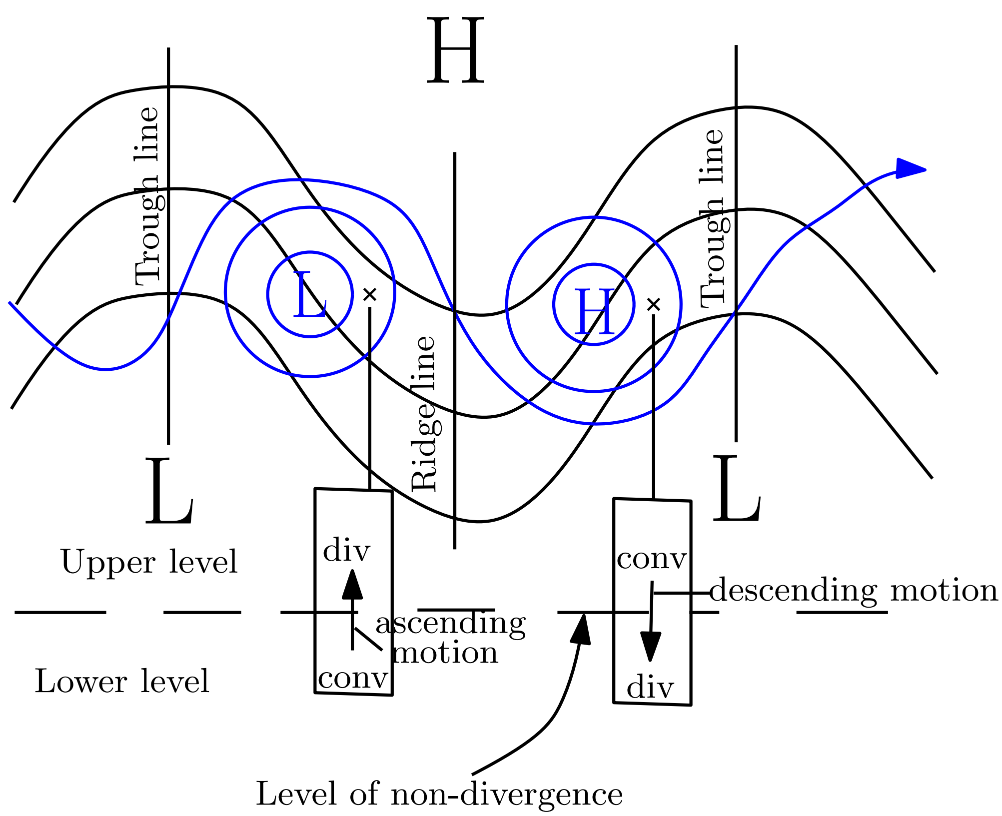
For low wind speeds and long wavelengths, the curvature term may be small, resulting in the wave moving westward as determined by the latitude effect. This effect is enforced when the wave amplitude is large.

Next, consider the case of equally spaced, concentric, circular isobars of a surface low and high pressure system in the Southern Hemisphere, which results in the curvature effect to be the same everywhere. However, the latitude effect will produce higher winds on the equatorward side.



The result of the latitude effect is convergence with rising pressure to the east (west) of the low (high) pressure and divergence with falling pressure to the west (east). Such systems are expected to move westward.

The idealized model



For the usual short-wave systems in which the curvature effect dominates the latitude effect, divergence is found east of the trough line and convergence ahead of the ridge line. East of the centre of the surface low, low-level convergence is found with divergence aloft, resulting in ascending motion. The opposite is found east of the ridge line where upper-level convergence is associated with low-level divergence east of the surface high, resulting in descending motion.

Consider the level of non-divergence shown in the figure. This is a level of transition from the positive to negative divergence, and vice versa. If this level is low in altitude, the high altitude pattern will predominate, and the system will move eastward. If this level is high in altitude, the low altitude pattern will predominate, and the system will move westward.

We have introduced here a classic theory qualitatively of the motion of pressure systems in mid-latitudes. Although this theory may reveal considerable quantitative agreement with synoptic experience, also over the Southern Hemisphere, we will develop and discuss quantitatively a set of equations that are less complicated than the full set of primitive equations of motion in order to describe extra-tropical weather systems. This set of equations represent the so-called quasi-geostrophic approximation. Why the theory is called quasi-geostrophic? It is because if the winds in mid-latitude systems were perfectly geostrophic, such winds never cross the isobars, and could thus not cause convergence into the low pressure system and therefore no vertical velocity. Since we know from observations that vertical motion does exist and are important for causing clouds and rain development in cyclones, the upward motion cannot be geostrophic. By including this ageostrophic flow into the set of equations that are otherwise totally geostrophic, the equations are said to be quasi-geostrophic, meaning partially geostrophic.

The quasi-geostrophic approximation

To show that for motions that are hydrostatic and nearly geostrophic, the 3-dimensional flow field is determined approximately by the **isobaric** distribution of geopotential $[\Phi(x, y, p, t)]$.

The use of the isobaric coordinate system simplifies the development of approximate prognostic and diagnostic equations.

Scale analysis in isobaric coordinates

Horizontal momentum equation	$\dfrac{D\overline{V}}{Dt} + f\overline{k} \times \overline{V} = -\overline{\nabla}\Phi$	(3.2) also (6.1)
Hydrostatic equation	$\dfrac{\partial\Phi}{\partial p} = -\alpha = -\dfrac{RT}{p}$	(3.27) also (6.2)
Continuity equation	$\overline{\nabla} \cdot \overline{V} + \dfrac{\partial\omega}{\partial p} = 0$	(3.5) also (6.3)
Thermodynamic energy equation	$\left(\dfrac{\partial}{\partial t} + \overline{V} \cdot \overline{\nabla}\right) T - S_p\omega = \dfrac{J}{c_p}$	(3.6) also (6.4)

Total derivative in (3.2):

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + \left(\overline{V} \cdot \overline{\nabla}\right)_p + \omega\frac{\partial}{\partial p} \qquad \left[\omega \equiv \frac{Dp}{Dt}\right] \tag{6.5}$$

From (3.6): $S_p \equiv -T\dfrac{\partial \ln\theta}{\partial p}$, static stability parameter $\left[S_p \sim 5 \times 10^{-4}\,\mathrm{K\,Pa^{-1}}\ \text{in mid-troposphere}\right]$

The above set of equations still contain several terms that are of secondary significance for mid-latitude synoptic-scale systems. They can be simplified further by

1) horizontal flow is nearly geostrophic

2) the magnitude of the ratio of vertical velocity to horizontal velocity is of the order 10^{-3}.

Separate the horizontal velocity into geostrophic and ageostrophic parts:

$$\overline{V} = \overline{V}_g + \overline{V}_a;\ \overline{V}_g \equiv \frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi \qquad \left[\overline{V}_a = \overline{V} - \overline{V}_g\right] \tag{6.7}$$

Regarding f_0: It is assumed that the meridional length scale (L) is small compared to the radius of the Earth so that the geostrophic wind (6.7) may be defined using a constant reference latitude value of the Coriolis parameter.

For the systems of interest

$$\left|\overline{V}_g\right| \gg \left|\overline{V}_a\right| \quad \text{or} \quad \frac{\left|\overline{V}_a\right|}{\left|\overline{V}_g\right|} \sim O(Ro), \text{ that is the same order of magnitude as the Rossby number}$$

$$\left(Ro \equiv \frac{U}{f_0 L} \sim 0.1 \text{ from Page 41 of Holton 4}\right)$$

Momentum can then be approximated to $O(Ro)$ by its geostrophic value, and the rate of change of momentum (or temperature) following the horizontal motion can be approximated to the same order by the rate of change following the geostrophic wind.

In equation (6.5):

1) $\overline{V}$ can be replaced by $\overline{V}_g$
2) the vertical advection which arises only from the ageostrophic flow can be neglected.

$$\therefore \frac{D\overline{V}}{Dt} \approx \frac{D_g \overline{V}_g}{Dt}$$

where

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \tag{6.8}$$

Note: Newton's second law; a form of the momentum equation:

$$\frac{D\overline{U}}{Dt} = -2\overline{\Omega} \times \overline{U} - \frac{\overline{1}}{\rho}\overline{\nabla} p + \overline{g} + \overline{F}_r \tag{2.8}$$

The dynamical effect of the variation of the Coriolis parameter with latitude needs to be retained in the Coriolis force term in the momentum equation. This variation can be approximated using a Taylor series:

$$f = f_0 + \left(\frac{df}{dy}\right)_{\phi_0} (y + y_0) + \text{higher order terms}$$

$$\beta \equiv \left(\frac{df}{dy}\right)_{\phi_0}, y = 0 \text{ at } \phi_0$$

This approximation is referred to as the: **mid-latitude β–plane** approximation

$$f = f_0 + \beta y \tag{6.9}$$

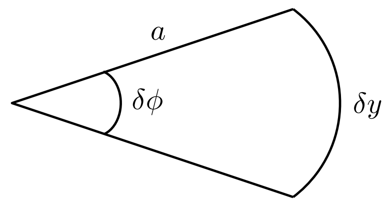
f_0 is the Coriolis parameter computed at a characteristic latitude, ϕ_0; the variable y measures the meridional distance from this latitude.

$$\beta = \left(\frac{df}{dy}\right)_{\phi_0}$$

$$= \frac{d}{dy}(2\Omega \sin\phi)_{\phi_0}$$

From the figure below:

$$\delta y = a\delta\phi$$
$$\therefore \frac{1}{\delta y} = \frac{1}{a}\frac{1}{\delta\phi}$$



$$\therefore \beta = \frac{1}{a}\frac{d}{d\phi}(2\Omega\sin\phi)_{\phi_0}$$
$$= \frac{2\Omega\cos\phi_0}{a}$$

The ratio of the terms on the right of (6.9):

$$\frac{\beta y}{f_0} \sim \frac{\beta L}{f_0} \sim \frac{2\Omega\cos\phi_0}{a} L \frac{1}{2\Omega\sin\phi_0}$$
$$= \frac{\cos\phi_0 L}{\sin\phi_0 a} \sim O(Ro) \ll 1 \quad \left[\text{Note: } \frac{\cos(-45^\circ)}{\sin(-45^\circ)} = -1 \text{ and } L \text{ is small compared to the radius of the Earth, } a\right]$$

$\therefore f_0 \gg \beta y$, which justifies letting the Coriolis parameter be a constant f_0 in the geostrophic approximation and using (6.9)

(6.1): $\dfrac{D\overline{V}}{Dt} + f\overline{k}\times\overline{V} + \overline{\nabla}\Phi = 0$ (the acceleration following the motion, the Coriolis force and the pressure gradient force are balanced)

Consider

$$f\overline{k}\times\overline{V} + \overline{\nabla}\Phi = (f_0+\beta y)\overline{k}\times(\overline{V}_g + \overline{V}_a) + \overline{\nabla}\Phi$$
$$= f_0\overline{k}\times\overline{V}_g + \beta y\overline{k}\times\overline{V}_g + f_0\overline{k}\times\overline{V}_a + \beta y\overline{k}\times\overline{V}_a - f_0\overline{k}\times\overline{V}_g$$
$$= f_0\overline{k}\times\overline{V}_a + \beta y\overline{k}\times\overline{V}_g + \beta y\overline{k}\times\overline{V}_a$$

Neglect the ageostrophic wind compared to the geostrophic wind in the term proportional to βy:

Atmospheric waves influenced by the beta (β) term are characterized as planetary waves (also called Rossby waves). These waves experience the curvature of a revolving planet through meridional changes in the Coriolis parameter. The so-called beta effect may be considered to be small when a synoptic-scale storm moves across only a small range of latitudes during its lifetime.

$$\therefore f\overline{k}\times\overline{V} + \overline{\nabla}\Phi \approx f_0\overline{k}\times\overline{V}_a + \beta y\overline{k}\times\overline{V}_g \tag{6.10}$$

The horizontal momentum equation i.t.o. geostrophic flow then becomes:

$$\frac{D_g \overline{V}_g}{Dt} = -f_0 \overline{k} \times \overline{V}_a - \beta y \overline{k} \times \overline{V}_g$$

and each of these terms is $O(Ro)$ compared to the pressure gradient force, and the neglected terms are $O(Ro^2)$ or smaller.

$$f_o \gg \beta y \quad \text{and} \quad |\overline{V}_g| \gg |\overline{V}_a|$$

$$\therefore \beta y \sim \frac{1}{10} f_o \qquad \therefore |\overline{V}_a| \sim \frac{1}{10} |\overline{V}_g|$$

$$\begin{aligned}
\frac{D_g \overline{V}_g}{Dt} &= - f_o \overline{k} x \overline{V_a} - \beta y \overline{k} x \overline{V_g} \\
&\sim - f_o \overline{k} x \left(\frac{1}{10} \overline{V_g}\right) - \frac{1}{10} f_o \overline{k} x \overline{V_g} \\
&\sim -\frac{1}{10} \left(f_o \overline{k}\, x \overline{V}_g\right) \\
&= -\frac{1}{10} \left(-\overline{\nabla} \Phi\right) \\
&= \frac{1}{10} \overline{\nabla} \Phi
\end{aligned}$$

$$(6.7): \quad f_o \overline{V}_g = \overline{k}\, x \overline{\nabla} \Phi$$

$$\therefore \overline{\nabla} \Phi = -f_o \overline{k}\, x \overline{V}_g$$

$$\text{since } \overline{k}\, x (\overline{k}\, x \overline{\nabla} \Phi) = -\overline{\nabla} \Phi$$

Next, $\overline{\nabla} \cdot \overline{V} = \overline{\nabla} \cdot (\overline{V}_g + \overline{V}_a) = \overline{\nabla} \cdot \overline{V}_g + \overline{\nabla} \cdot \overline{V}_a$

Since $\overline{V}_g = \dfrac{1}{f_0} \overline{k} \times \overline{\nabla} \Phi$ is non-divergent, $\overline{\nabla} \cdot \overline{V}_g = 0$

$$\therefore \overline{\nabla} \cdot \overline{V} = \overline{\nabla} \cdot \overline{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

$$(6.3) : \overline{\nabla} \cdot \overline{V} + \frac{\partial \omega}{\partial p} = 0 \implies \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{6.12}$$

(6.12) means that ω is determined only by the ageostrophic part of the wind field.

The thermodynamic energy equation (6.4): $\left(\dfrac{\partial}{\partial t} + \overline{V} \cdot \overline{\nabla}\right) T - S_p \omega = \dfrac{J}{c_p}$

However, the **horizontal advection** can be approximated by the geostrophic value

$$\therefore \left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right) T - S_p \omega = \frac{J}{c_p}$$

The **vertical advection** is not neglected and forms part of the adiabatic heating and cooling term. This term must be retained because the static stability is usually large enough on the synoptic scale so that the adiabatic heating/cooling due to vertical motion is of the same order as the horizontal temperature advection.

Simplifying the adiabatic heating and cooling term: Divide the total temperature field, T_{tot}, into a basic state (standard atmosphere) portion that depends only on pressure, $T_0(p)$, plus a deviation from the basic state, $T(x, y, p, t)$.

$$T_{tot}(x, y, p, t) = \underbrace{T_0(p)}_{\text{Basic state}} + \overbrace{T(x, y, p, t)}^{\text{Deviation from the basic state}}$$

Static stability parameter in the isobaric system

$$S_p \equiv -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \tag{3.7}$$

$$S_p = -T\frac{\partial \ln \theta}{\partial p} = -T_0 \frac{\partial \ln \theta}{\partial p}$$

because $\left|\dfrac{dT_0}{dp}\right| \gg \left|\dfrac{\partial T}{\partial p}\right|$

θ_0 is the potential temperature that corresponds to the basic state temperature T_0, which is only a function of p $[T_0 = T_0(p)]$.

$$\therefore \frac{\partial \ln \theta_0}{\partial p} = \frac{d \ln \theta_0}{dp}$$

$$\begin{aligned}\therefore S_p = -T_0 \frac{d \ln \theta_0}{dp} &= -T_0 \frac{d \ln \theta_0}{dp}\left(\frac{p}{R}\right)\left(\frac{R}{p}\right) \\ &= -\frac{RT_0}{p}\frac{d \ln \theta_0}{dp}\left(\frac{p}{R}\right)\end{aligned}$$

$$\sigma \equiv -\frac{RT_0}{p}\frac{d \ln \theta_0}{dp}$$

$$\therefore S_p = \frac{\sigma p}{R}$$

$$\therefore \left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right) T - \left(\frac{\sigma p}{R}\right) \omega = \frac{J}{c_p}$$

(6.2): $\dfrac{\partial \Phi}{\partial p} = -\dfrac{RT}{p} \implies T = -\dfrac{p}{R}\dfrac{\partial \Phi}{\partial p}$

$$\therefore \left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right)\left(\frac{p}{R}\right)\left(\frac{\partial \Phi}{\partial p}\right) - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right)\left(\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{R}{p}\frac{J}{c_p} = \frac{\kappa J}{p} \qquad \left[\kappa \equiv \frac{R}{c_p}\right] \tag{6.13b}$$

The quasi-geostrophic equations form a complete set in the dependent variables $\Phi, \overline{V}_g, \overline{V}_a$, and ω.

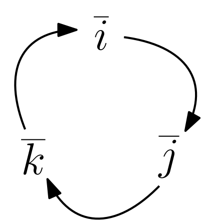
$$\overline{V}_g = \frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi \tag{6.7}$$

$$\frac{D_g \overline{V}_g}{Dt} = -f_0 \overline{k} \times \overline{V}_a - \beta y \overline{k} \times \overline{V}_g \tag{6.11}$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{6.12}$$

$$\left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \tag{6.13b}$$

The quasi-geostrophic vorticity equation



$$\overline{V}_g = \frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi \tag{6.7}$$

$$\begin{aligned} f_0 \overline{V}_g = f_0(u_g \overline{i} + v_g \overline{j}) &= \overline{k} \times \left(\frac{\partial \Phi}{\partial x}\overline{i} + \frac{\partial \Phi}{\partial y}\overline{j}\right) \\ &= \frac{\partial \Phi}{\partial x}\overline{j} - \frac{\partial \Phi}{\partial y}\overline{i} \end{aligned}$$

$$\therefore f_0 v_g = \frac{\partial \Phi}{\partial x}, \quad f_0 u_g = -\frac{\partial \Phi}{\partial y} \tag{6.14}$$

Geostrophic vorticity

$$
\begin{aligned}
\zeta_g = \overline{k} \cdot \overline{\nabla} \times \overline{V}_g &= \overline{k} \begin{vmatrix} \bar{i} & \bar{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u_g & v_g & 0 \end{vmatrix} \\
&= \overline{k} \cdot \left[\overline{k} \left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) \right] \\
&= \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \zeta_g &= \frac{\partial}{\partial x} \left(\frac{1}{f_0} \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{1}{f_0} \left(-\frac{\partial \Phi}{\partial y} \right) \right) \\
&= \frac{1}{f_0} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{1}{f_0} \nabla^2 \Phi
\end{aligned} \tag{6.15}
$$

This equation can be used to determine $\zeta_g(x, y)$ from a known field $\Phi(x, y)$.

It can also be solved by inverting the Laplacian operator to determine Φ from a known distribution of ζ_g, provided that suitable conditions on Φ are specified on the boundaries of the region in question.

Vorticity is a useful forecast diagnostic: if the evolution of the vorticity can be predicted, then inversion of (6.15) yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Note: The Laplacian of a function tends to be a maximum where the function itself is a minimum...

$$
\zeta_g = \frac{1}{f_0} \nabla^2 \Phi
$$

It will be shown later in the course that $\nabla^2 \Phi \propto -\Phi$.

In Northern Hemisphere:

$$
\begin{aligned}
\frac{1}{f_0} \nabla^2 \Phi \propto -\Phi \quad &\text{since } f_0 > 0 \\
\therefore \zeta_g \propto -\Phi
\end{aligned}
$$

$\Longrightarrow$ positive vorticity implies low values of geopotential, and vice versa.
At ridge Φ is a maximum, thus $\zeta_g < 0$
At trough Φ is a minimum, thus $\zeta_g > 0$

In Southern Hemisphere:

$$
\begin{aligned}
\frac{1}{f_0} \nabla^2 \Phi \propto \Phi \quad &\text{since } f_0 < 0 \\
\therefore \zeta_g \propto \Phi
\end{aligned}
$$

$\Longrightarrow$ positive vorticity implies low values of geopotential, and vice versa.

At ridge Φ is a maximum, thus $\zeta_g > 0$

At trough Φ is a minimum, thus $\zeta_g < 0$

The quasi-geostrophic vorticity equation can be obtained from the quasi-geostrophic momentum equation (6.11):

$$\frac{D_g \overline{V}_g}{Dt} = -f_0 \overline{k} \times \overline{V}_a - \beta y \overline{k} \times \overline{V}_g$$

$$\begin{aligned}\frac{D_g}{Dt}(u_g \overline{i} + v_g \overline{j}) &= -f_0 \overline{k} \times (u_a \overline{i} + v_a \overline{j}) - \beta y \overline{k} \times (u_g \overline{i} + v_g \overline{j}) \\ &= -f_0 u_a \overline{j} - f_0(-v_a \overline{i}) - \beta y u_g \overline{j} - \beta y(-v_g \overline{i})\end{aligned}$$

$$\therefore \; \frac{D_g}{Dt} u_g = f_0 v_a + \beta y v_g$$

$$\& \; \frac{D_g}{Dt} v_g = -f_0 u_a - \beta y u_g$$

$$\therefore \; \frac{D_g}{Dt} u_g - f_0 v_a - \beta y v_g = 0 \tag{6.16}$$

$$\& \; \frac{D_g}{Dt} v_g + f_0 u_a + \beta y u_g = 0 \tag{6.17}$$

$\frac{\partial}{\partial x}(6.17) - \frac{\partial}{\partial y}(6.16)$:

$$\frac{\partial}{\partial y}\left(\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} - f_0 v_a - \beta y v_g\right) = 0$$

$$\begin{aligned}\frac{\partial^2 u_g}{\partial y \partial t} + \frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x} + u_g \frac{\partial^2 u_g}{\partial x \partial y} + \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y} + v_g \frac{\partial^2 u_g}{\partial y^2} - v_a \overbrace{\frac{\partial f_0}{\partial y}}^{=0} - f_0 \frac{\partial v_a}{\partial y} \\ - \frac{\partial \beta}{\partial y} y v_g - \beta \underbrace{\frac{\partial y}{\partial y}}_{=1} v_g - \beta y \frac{\partial v_g}{\partial y} = 0\end{aligned} \tag{1}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} + f_0 u_a + \beta y u_g\right) = 0$$

$$\begin{aligned}\frac{\partial^2 v_g}{\partial x \partial t} + \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} + u_g \frac{\partial^2 v_g}{\partial x^2} + \frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y} + v_g \frac{\partial^2 v_g}{\partial x \partial y} + u_a \overbrace{\frac{\partial f_0}{\partial x}}^{=0} + f_0 \frac{\partial u_a}{\partial x} \\ + \frac{\partial \beta}{\partial x} y u_g + \beta \underbrace{\frac{\partial y}{\partial x}}_{=0} u_g + \beta y \frac{\partial u_g}{\partial x} = 0\end{aligned} \tag{2}$$

$(2) - (1)$:

$$
\begin{aligned}
&\frac{\partial^2 v_g}{\partial x \partial t} - \frac{\partial^2 u_g}{\partial y \partial t} + u_g \frac{\partial^2 v_g}{\partial x^2} - u_g \frac{\partial^2 u_g}{\partial x \partial y} + v_g \frac{\partial^2 v_g}{\partial x \partial y} - v_g \frac{\partial^2 u_g}{\partial y^2} + f_0 \frac{\partial u_a}{\partial x} + f_0 \frac{\partial v_a}{\partial y} \\
&+ \frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y} \\
&+ \frac{\partial \beta}{\partial x} y u_g + \frac{\partial \beta}{\partial y} y v_g + \beta v_g + \beta y \frac{\partial u_g}{\partial x} + \beta y \frac{\partial v_g}{\partial y} = 0 \\
\therefore\ &\frac{\partial}{\partial t}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + u_g \frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + v_g \frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) \\
&+ \frac{\partial u_g}{\partial x}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \frac{\partial v_g}{\partial y}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) \\
&+ \text{“}\beta \text{ terms”} = 0 \\
\therefore\ &\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) \\
&+ \zeta_g\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) \\
&+ 0 + 0 + \beta v_g + \beta y\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = 0
\end{aligned}
$$

But the divergence of the geostrophic wind vanishes:

$$
\begin{aligned}
\overline{\nabla} \cdot \overline{V}_g &= 0 \\
\therefore\ \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} &= 0 \\
\implies \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)\zeta_g &= -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \\
\therefore\ \frac{D_g}{Dt}\zeta_g &= -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \qquad (6.18)
\end{aligned}
$$

Take note

$$
\begin{aligned}
\frac{D_g f}{Dt} &= \underbrace{\frac{\partial f}{\partial t}}_{=0} + u_g \underbrace{\frac{\partial f}{\partial x}}_{=0} + v_g \frac{\partial f}{\partial y} \qquad [f = f(y)] \\
&= 0 + \overline{V}_g \cdot \overline{\nabla} f = \beta v_g \\
\therefore\ \frac{\partial \zeta_g}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\zeta_g &= -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \overline{V}_g \cdot \overline{\nabla} f
\end{aligned}
$$

From (6.12): $\dfrac{\partial u_a}{\partial x} + \dfrac{\partial v_a}{\partial y} = -\dfrac{\partial \omega}{\partial p}$

$$
\begin{aligned}
\therefore\ \frac{\partial \zeta_g}{\partial t} &= -\overline{V}_g \cdot \overline{\nabla}\zeta_g - f_0\left(-\frac{\partial \omega}{\partial p}\right) - \overline{V}_g \cdot \overline{\nabla} f \\
&= -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \qquad (6.19)
\end{aligned}
$$

In words: The local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect).

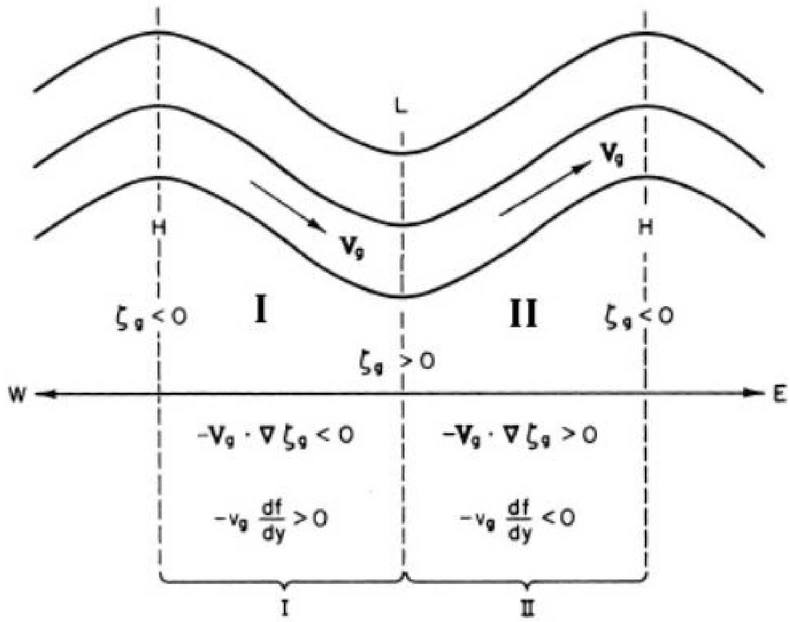
$$\text{Vorticity tendency due to vorticity advection: } -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f)$$
$$= -\overline{V}_g \cdot \overline{\nabla}\zeta_g - \beta v_g$$

$\overline{V}_g \cdot \overline{\nabla}\zeta_g$: geostrophic advection of relative vorticity

βv_g : geostrophic advection of planetary vorticity

For disturbances in the westerlies, these two effects tend to have opposite signs.

Consider the figure for an idealized 500hPa flow in the **Northern Hemisphere**.



In **region I**, upstream of the 500hPa trough, the geostrophic wind is directed from the relative vorticity minimum at the ridge towards the relative vorticity maximum at the trough.

$$\therefore \overline{V}_g \cdot \overline{\nabla}\zeta_g > 0 \implies -\overline{V}_g \cdot \overline{\nabla}\zeta_g < 0$$

At the same time $v_g < 0$ in **region I** because it is directed southwards.

Take note that $\beta = 2\Omega \cos\phi_0 / a > 0$ in both hemispheres.

$$\therefore \beta v_g < 0 \implies -\beta v_g > 0$$

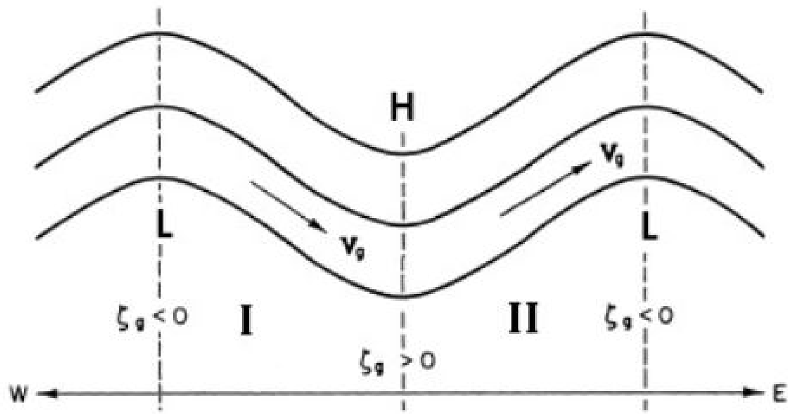
We now have in region I that the:

1) advection of relative vorticity tends to **decrease** the local vorticity
2) advection of planetary vorticity tends to **increase** the local vorticity.

The same arguments can be applied for region II.

Therefore, advection of relative vorticity tends to move the vorticity pattern and hence the troughs and ridges eastward (downstream). However, advection of planetary vorticity tends to move the troughs and ridges westward against the advecting wind field.

The net effect of advection on the evolution of the vorticity pattern depends on which type of vorticity advection dominates.



Consider the schematic of the $500\mathrm{hPa}$ geopotential field in the Southern Hemisphere above.

The advection of the absolute vorticity by the geostrophic wind:

$$-\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) = -\overline{V}_g \cdot \overline{\nabla}\zeta_g - \beta v_g$$

Region I: Advection of relative vorticity is positive because we are going from $\zeta_g < 0$ at the trough to $\zeta_g >$ at the ridge.

$$\therefore \overline{V}_g \cdot \overline{\nabla}\zeta_g > 0 \implies -\overline{V}_g \cdot \overline{\nabla}\zeta_g < 0$$

We have shown that $\beta > 0$. However, in the region v_g points southwards. Therefore, $v_g < 0$.

$$\therefore \beta v_g < 0 \implies -\beta v_g > 0$$

Region II: $-\overline{V}_g \cdot \overline{\nabla}\zeta_g > 0$, because advection of relative vorticity is negative and $-\beta v_g < 0$ because $v_g > 0$.

Consider an idealised geopotential distribution on a mid-latitude β-plane of the form

$$\Phi(x, y) = \Phi_0 - f_0 U y + f_0 A \sin kx \cos ly$$

Φ_0, a constant zonal speed U, and amplitude A depend only on pressure. Wave numbers k and l are defined as $k = 2\pi/L_x$ and $l = 2\pi/L_y$. L_x and L_y are respectively the wavelengths in the x and y directions. y in the geopotential distribution equation is given by $a(\phi - \phi_0)$, with a the radius of the Earth and ϕ_0 the latitude at which f_0 is evaluated.

$$\begin{aligned} \text{For } \phi - \phi_0 &= 6^\circ,\ y = 6.67 \times 10^5\,\mathrm{m} \\ &= 3^\circ,\ y = 3.34 \times 10^5\,\mathrm{m} \\ &= 1^\circ,\ y = 1.11 \times 10^5\,\mathrm{m} \\ &= 10^\circ,\ y = 1.11 \times 10^6\,\mathrm{m} \end{aligned}$$

Therefore, for $10°$ displacement in y, y is approximately equal to the length scale, L, of 10^6 m.

$$\begin{aligned} u_g &= -\frac{1}{f_0}\frac{\partial}{\partial y}\left(\Phi_0 - f_0 U y + f_0 A \sin kx \cos ly\right) \\ &= U - A \sin kx(-l \sin ly) = U + lA \sin kx \sin ly \end{aligned}$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = \frac{1}{f_0}\left(f_0 kA \cos kx \cos ly\right) = kA \cos kx \cos ly$$

$$\zeta_g = \frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi$$

$$\frac{\partial}{\partial x}\left(f_0 kA \cos kx \cos ly\right) = -f_0 k^2 A \sin kx \cos ly$$

$$\frac{\partial}{\partial y}\left(-f_0 U - f_0 lA \sin kx \sin ly\right) = -f_0 l^2 A \sin kx \cos ly$$

$$\begin{aligned} \therefore \zeta_g &= \frac{1}{f_0}\left(-f_0 k^2 - f_0 l^2\right) A \sin kx \cos ly \\ &= -(k^2 + l^2) A \sin kx \cos ly \end{aligned}$$

Advection of relative vorticity:

$$\begin{aligned} -\overline{V}_g \cdot \overline{\nabla}\zeta_g &= -\left(u_g \bar{i} + v_g \bar{j}\right) \cdot \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j}\right)\zeta_g \\ &= -u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} \\ &= -u_g(-(k^2 + l^2)Ak \cos kx \cos ly) - v_g(-(k^2 + l^2)Al \sin kx(-\sin ly)) \\ &= -u_g(-(k^2 + l^2)v_g) - v_g((k^2 + l^2)(u_g - U)) \\ &= u_g v_g(k^2 + l^2) - u_g v_g(k^2 + l^2) + v_g U(k^2 + l^2) \\ &= v_g U(k^2 + l^2) = Ak \cos kx \cos ly U(k^2 + l^2) \\ &= kU(k^2 + l^2)A \cos kx \cos ly \end{aligned}$$

Advection of planetary vorticity:

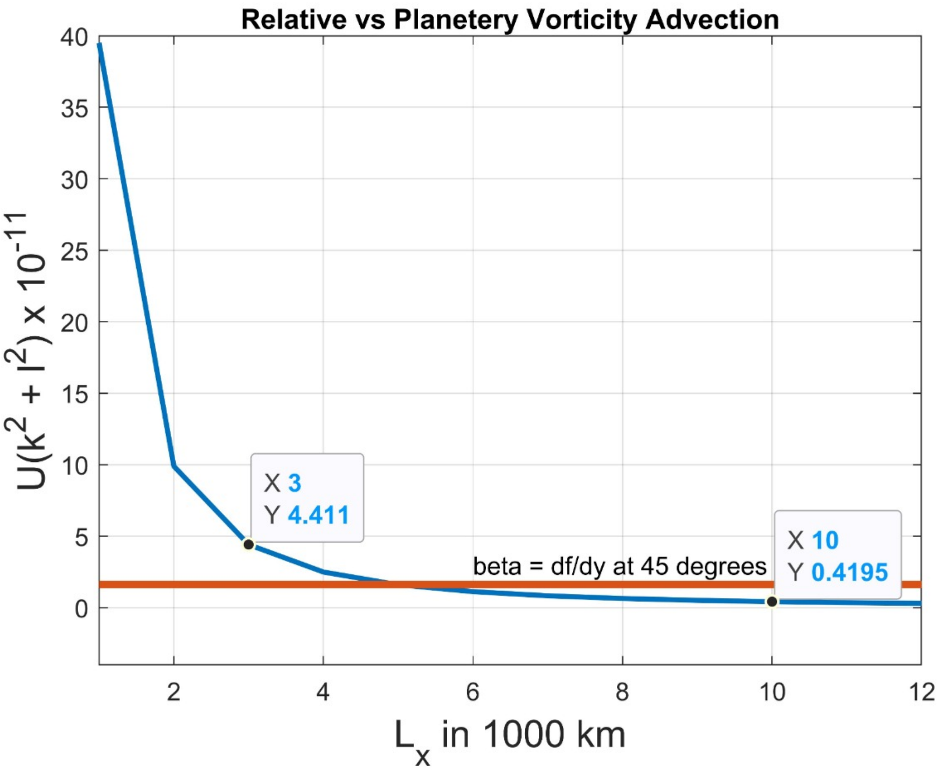
$$-v_g \frac{df}{dy} = -v_g \beta = -\beta Ak \cos kx \cos ly \quad [\beta = 2\Omega \cos \phi_0 / a]$$

Advection of absolute vorticity:

$$\begin{aligned} -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) &= kU(k^2 + l^2)A \cos kx \cos ly - \beta Ak \cos kx \cos ly \\ &= kA \cos kx \cos ly(U(k^2 + l^2) - \beta) \end{aligned}$$

$k = \frac{2\pi}{L_x}$, $l = \frac{2\pi}{L_y}$ with L_x and L_y the wavelengths in the x and y directions, respectively.

Consider $l = \frac{\pi}{2} \times 10^{-7}\,\mathrm{m}^{-1}$ for fixed L_y wavelengths. However, we want to determine the effect of L_x on the advection of both relative and planetary vorticity, and so wavenumber k varies with a range of L_x (i.e., $1000\,\mathrm{km}$ to $12000\,\mathrm{km}$). We therefore need to evaluate $U(k^2 + l^2)$ against β as shown in the figure below.



Take note that the term representing the advection of relative vorticity ($U(k^2 + l^2)$) at $L_x = 3000\,\mathrm{km}$ is about ten times larger than the value at $L_x = 10000\,\mathrm{km}$. This result implies that relative vorticity advection is multiple times larger than planetary vorticity advection at $3000\,\mathrm{km}$ where there is a clear exponential inflection on the figure.

By considering a simplified version of an idealised geopotential distribution, a similar result is obtained.

$$\Phi(x, y, p) = \Phi_0(p) - f_0 U_0 y \sin\left(\frac{\pi p}{p_0}\right) + f_0 A \sin kx$$

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi = -\frac{1}{f_0}\left(-f_0 U_0 \sin\left(\frac{\pi p}{p_0}\right)\right) = U_0 \sin\left(\frac{\pi p}{p_0}\right)$$

$$v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\Phi = \frac{1}{f_0}\left(f_0 A k \cos kx\right) = Ak\cos kx$$

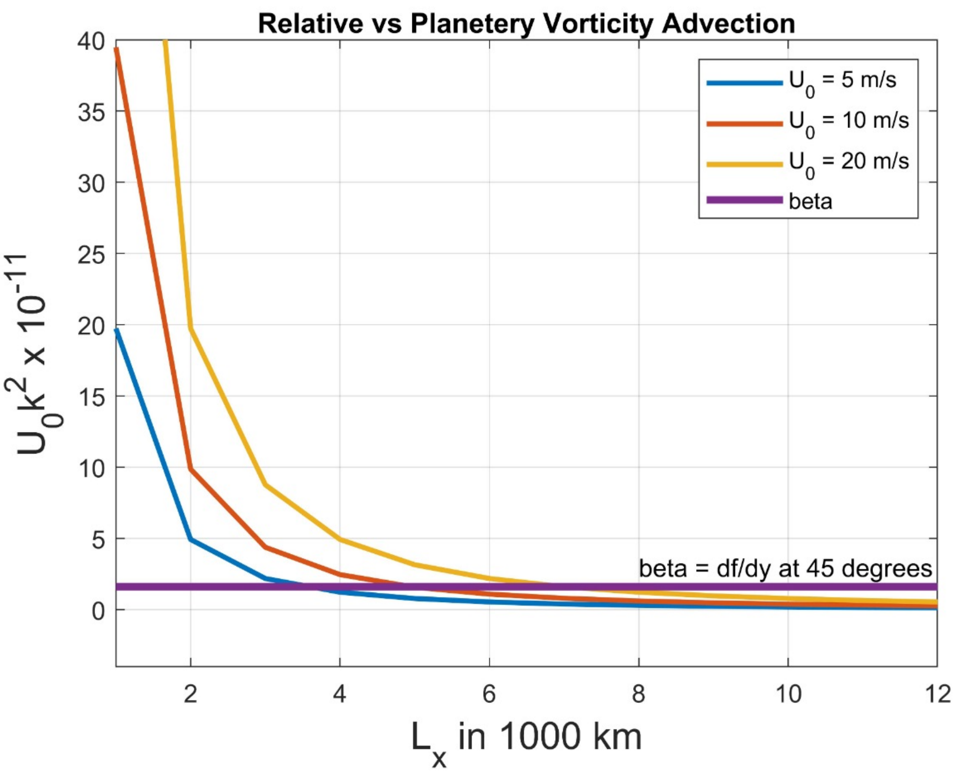
$$\begin{aligned}\zeta_g &= \frac{1}{f_0}\left(\frac{\partial}{\partial x}\left(f_0 A k \cos kx\right) + \frac{\partial}{\partial y}\left(-f_0 U_0 \sin\left(\frac{\pi p}{p_0}\right)\right)\right) \\ &= \frac{1}{f_0}\left(-f_0 A k^2 \sin kx\right) = -k^2 A \sin kx\end{aligned}$$

$$-\overline{V}_g \cdot \overline{\nabla}\zeta_g = k^2 U_0 \sin\left(\frac{\pi p}{p_0}\right) Ak\cos kx$$

$$\therefore -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) = \left(k^2 U_0 \sin\left(\frac{\pi p}{p_0}\right) - \beta\right) Ak\cos kx$$
$$= \left(k^2 U_0 - \beta\right) Ak\cos kx$$

when $p_0 = 1000\text{hPa}$ and $p = 500\text{hPa}$.

Here we also show the results of having different values of U_0, the constant zonal speed. Clearly, the strength of a constant zonal wind will affect the wave lengths of short-wave systems, but will have a minimal affect on the wavelengths of Rossby waves



Exercise 1: Suppose that on the 500hPa surface of the schematic above, the relative vorticity at a certain location at $45°$S latitude is increasing at a rate of $3 \times 10^{-6}\,\text{s}^{-1}$ per 3 hours. The wind is from the northwest at $20\,\text{m}\,\text{s}^{-1}$ and the relative vorticity increases towards the southeast at a rate of $4 \times 10^{-6}\,\text{s}^{-1}$ per $100\,\text{km}$. Use the quasi-geostrophic vorticity equation to estimate the horizontal divergence at this location on a β-plane.

Make use of the following assumptions:

1. The constant Coriolis parameter is equal to $-10^{-4}\,\text{s}^{-1}$ in the Southern Hemisphere
2. β is approximated by $10^{-11}\,\text{m}^{-1}\,\text{s}^{-1}$
3. The following relationship is valid for natural coordinates: $\overline{V}_g \cdot \overline{\nabla}\zeta_g \sim v\dfrac{\partial \zeta_g}{\partial s}$, where s is the distance along the curve (500hPa contour)

Solution:

$$
\begin{aligned}
\frac{\partial \zeta_g}{\partial t} &= -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) + f_0(-\overline{\nabla} \cdot \overline{V}) \\
\therefore f_0 \overline{\nabla} \cdot \overline{V} &= -\frac{\partial \zeta_g}{\partial t} - \overline{V}_g \cdot \overline{\nabla}\zeta_g - v_g \frac{\partial f}{\partial y} \\
&= -\frac{\partial \zeta_g}{\partial t} - v\frac{\partial \zeta_g}{\partial s} - v_g \beta
\end{aligned}
$$

$$\frac{\partial \zeta_g}{\partial t} = \frac{3 \times 10^{-6}\,\mathrm{s}^{-1}}{(3 \times 3600)\,\mathrm{s}} = 2.778 \times 10^{-10}\,\mathrm{s}^{-2}$$

$$v\frac{\partial \zeta_g}{\partial s} = (20\,\mathrm{m\,s}^{-1})\left(\frac{4 \times 10^{-6}\,\mathrm{s}^{-1}}{100\,000\,\mathrm{m}}\right) = 8 \times 10^{-10}\,\mathrm{s}^{-2}$$

$$
\begin{aligned}
(20\,\mathrm{m\,s}^{-1})^2 &= {u_g}^2 + {v_g}^2, \quad u_g = v_g \text{ (Pythagoras)} \\
\therefore v_g &= \pm\left(\frac{20^2}{2}\right)^{\frac{1}{2}} \\
\therefore v_g &= -14.14\,\mathrm{m\,s}^{-1} \quad (\text{since } v_g < 0) \\
\therefore v_g \beta &= -14.14\,\mathrm{m\,s}^{-1}(10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}) \\
&= -1.414 \times 10^{-10}\,\mathrm{s}^{-2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \overline{\nabla} \cdot \overline{V} &= -{f_0}^{-1}\left(\frac{\partial \zeta_g}{\partial t} + v\frac{\partial \zeta_g}{\partial s} + v_g\beta\right) \\
&= -(-10^{-4}\,\mathrm{s}^{-1})^{-1}\left(2.778 \times 10^{-10} + 8 \times 10^{-10} - 1.414 \times 10^{-10}\right)\mathrm{s}^{-2} \\
&= 9.364 \times 10^{-6}\,\mathrm{s}^{-1}, \text{ divergence}
\end{aligned}
$$

Exercise 2: Consider the following expression for the geopotential field:

$$\Phi = \Phi_0(p) + cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right] + k^{-1}\sin k(x - ct)\right\}$$

is a function of p alone, c is a constant speed, k a zonal wave number, and $p_0 = 1000\mathrm{hPa}$.

Consider the following two assumptions:

1. Only consider the dominating vorticity advection (either planetary or relative) term applicable to short-wave systems
2. Geostrophic relative vorticity only varies between trough and ridge axes in the x-direction

Use the quasi-geostrophic vorticity equation to show that the horizontal divergence field consistent with this geopotential field can be expressed as:

$$(f_0)^{-1}(ck)^2\cos\left(\frac{\pi p}{p_0}\right)\cos k(x - ct)$$

<u>Solution</u>: From Exercise 1: $\overline{\nabla}\cdot\overline{V} = -{f_0}^{-1}\left(\frac{\partial}{\partial t} + \overline{V}_g\cdot\overline{\nabla}\right)(\zeta_g + f)$

$$\begin{aligned} u_g &= -\frac{1}{f_0}\frac{\partial\Phi}{\partial y} \\ v_g &= \frac{1}{f_0}\frac{\partial\Phi}{\partial x} \\ \zeta_g &= \frac{1}{f_0}\nabla^2\Phi \end{aligned}$$

$$\begin{aligned} -f_0\overline{\nabla}\cdot\overline{V} &= \left(\frac{\partial}{\partial t} + \overline{V}_g\cdot\overline{\nabla}\right)(\zeta_g + f) = \frac{\partial\zeta_g}{\partial t} + \frac{\partial f}{\partial t} + \overline{V}_g\cdot\overline{\nabla}\zeta_g + \overline{V}_g\cdot\overline{\nabla}f \\ &= \frac{\partial\zeta_g}{\partial t} + (u_g\bar{i} + v_g\bar{j})\cdot\left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j}\right)\zeta_g + (u_g\bar{i} + v_g\bar{j})\cdot\left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j}\right)f \\ &= \frac{\partial\zeta_g}{\partial t} + u_g\frac{\partial\zeta_g}{\partial x} + v_g\frac{\partial\zeta_g}{\partial y} + v_g\frac{\partial f}{\partial y} \end{aligned}$$

We are considering short-wave systems, which means planetary vorticity advection is dominated by relative vorticity advection, thus $v_g\beta \sim 0$.

Also according to idealized $500\mathrm{hPa}$ geopotential field, ζ_g only varies between trough and ridge axes in the x-direction, therefore $v_g\frac{\partial\zeta_g}{\partial y} = 0$.

$$\begin{aligned} \therefore -f_0\overline{\nabla}\cdot\overline{V} &= \frac{\partial\zeta_g}{\partial t} + u_g\frac{\partial\zeta_g}{\partial x} \\ \therefore \overline{\nabla}\cdot\overline{V} &= -{f_0}^{-1}\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x}\right)\zeta_g \end{aligned}$$

$$\begin{aligned} \zeta_g &= \frac{1}{f_0}\nabla^2\Phi \\ \implies f_0\zeta_g &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right] + \frac{1}{k}\sin k(x - ct)\right\}\right) \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right] + \frac{1}{k}\sin k(x - ct)\right\}\right)\right) \\ &= \frac{\partial}{\partial x}\left(cf_0\frac{1}{k}k\cos k(x - ct)\right) \\ &= -cf_0 k\sin k(x - ct) \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(cf_0\left\{-y\cos\left(\frac{\pi p}{p_0}\right) - y + \frac{1}{k}\sin k(x - ct)\right\}\right)\right) \\ &= \frac{\partial}{\partial y}\left(cf_0\left(-\cos\left(\frac{\pi p}{p_0}\right) - 1\right)\right) \\ &= 0 \end{aligned}$$

$$f_0\zeta_g = -cf_0 k \sin k(x-ct)$$
$$\therefore \zeta_g = -ck \sin k(x-ct)$$

$$\begin{aligned}
u_g &= -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \\
\implies -f_0 u_g &= \frac{\partial}{\partial y}\left(cf_0\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right)+1\right]+\frac{1}{k}\sin k(x-ct)\right\}\right) \\
&= cf_0\left(-\left[\cos\left(\frac{\pi p}{p_0}\right)+1\right]\right) \\
\therefore u_g &= c\left(\cos\left(\frac{\pi p}{p_0}\right)+1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\zeta_g &= \frac{\partial}{\partial t}(-ck\sin k(x-ct)) \\
&= -ck(-ck\cos k(x-ct)) \\
&= c^2k^2\cos k(x-ct)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\zeta_g &= \frac{\partial}{\partial x}(-ck\sin k(x-ct)) \\
&= -ck(k\cos k(x-ct)) \\
&= -ck^2\cos k(x-ct)
\end{aligned}$$

$$\begin{aligned}
\therefore \overline{\nabla}\cdot\overline{V} &= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)+\left(c\left(\cos\left(\frac{\pi p}{p_0}\right)+1\right)\right)\times(-ck^2\cos k(x-ct))\right) \\
&= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)+\left(c\cos\left(\frac{\pi p}{p_0}\right)+c\right)\times(-ck^2\cos k(x-ct))\right) \\
&= -\frac{1}{f_0}\left(c^2k^2\cos k(x-ct)-c^2k^2\cos\left(\frac{\pi p}{p_0}\right)\cos k(x-ct)-c^2k^2\cos k(x-ct)\right) \\
&= \frac{c^2k^2}{f_0}\cos\left(\frac{\pi p}{p_0}\right)\cos k(x-ct)
\end{aligned}$$

Exercise 3: Suppose that on the $500\mathrm{hPa}$ surface the relative vorticity at a location just left of the ridge line in the figure used in Exercise 1, at the 45°S latitude (where the Coriolis parameter can be considered to be a constant value of $-10^{-4}\,\mathrm{s}^{-1}$ in the Southern Hemisphere) is increasing at a rate of $3.6\times10^{-6}\,\mathrm{s}^{-1}$ per hour. The wind is, for all practical purposes, blowing directly from the west above the location (negligible north-south component) at $20\,\mathrm{m\,s}^{-1}$ and the relative vorticity increases toward the east at a rate of $4\times10^{-6}\,\mathrm{s}^{-1}$ per $100\,\mathrm{km}$. Use the quasi-geostrophic vorticity equation to estimate the horizontal divergence at this location on a β-plane. This is a short-wave system.

Solution:

$$\begin{aligned}
\frac{\partial \zeta_g}{\partial t} &= -\overline{V}_g\cdot\overline{\nabla}(\zeta_g+f)+f_0\frac{\partial\omega}{\partial p} \\
&= -(u_g\bar{i}+v_g\bar{j})\cdot\left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}\right)(\zeta_g+f)+f_0\frac{\partial\omega}{\partial p}
\end{aligned}$$

Since we can ignore advection of planetary vorticity,

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} + f_0 \frac{\partial \omega}{\partial p}$$

Since the wind at the location is blowing from the west, $v_g = 0$

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} + f_0 \frac{\partial \omega}{\partial p}$$

$$\therefore \frac{\partial \omega}{\partial p} = {f_0}^{-1} \left(\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} \right)$$

$$\begin{aligned}
\frac{\partial \zeta_g}{\partial t} &> 0 \text{ (increasing at a rate of } 3.6 \times 10^{-6}\,\mathrm{s}^{-1}\text{)} \\
u_g &> 0 \text{ (wind from the west)} \\
\frac{\partial \zeta_g}{\partial x} &> 0 \text{ (vorticity increases per distance, and location is in Region I)}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{\partial \omega}{\partial p} &= \left(-10^{-4}\,\mathrm{s}^{-1}\right)^{-1} \left(\frac{3.6 \times 10^{-6}\,\mathrm{s}^{-1}}{1 \times 60 \times 60\,\mathrm{s}} + 20\,\mathrm{m\,s}^{-1}\, \frac{4 \times 10^{-6}\,\mathrm{s}^{-1}}{10^3\,\mathrm{m}} \right) \\
&= -10^4\,\mathrm{s} \left(\frac{36 \times 10^{-7}}{36 \times 10^2}\,\mathrm{s}^{-2} + 8 \times 10^{-10}\,\mathrm{s}^{-2} \right) \\
&= -10^4 \times \left(10^{-9}\,\mathrm{s}^{-1} + 8 \times 10^{-10}\,\mathrm{s}^{-1}\right) \\
&= -10^4 \times \left(10 \times 10^{-10} + 8 \times 10^{-10}\right)\mathrm{s}^{-1} \\
&= -1.8 \times 10^{-5}\,\mathrm{s}^{-1}, \text{ divergence since } -\frac{\partial \omega}{\partial p} = \overline{\nabla} \cdot \overline{V}
\end{aligned}$$

Quasi-geostrophic prediction

The geostrophic vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \tag{6.19}$$

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi \tag{6.15}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{1}{f_0} \nabla^2 \Phi \right) &= -\overline{V}_g \cdot \overline{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p} \\
\frac{1}{f_0} \nabla^2 \frac{\partial \Phi}{\partial t} &= -\overline{V}_g \cdot \overline{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p}
\end{aligned}$$

Defining the geopotential tendency $\chi \equiv \dfrac{\partial \Phi}{\partial t}$

$$\therefore \frac{1}{f_0} \nabla^2 \chi = -\overline{V}_g \cdot \overline{\nabla} \left(\frac{1}{f_0} \nabla^2 \Phi + f \right) + f_0 \frac{\partial \omega}{\partial p} \tag{6.21}$$

Since $\overline{V}_g = \dfrac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi$, the right-hand side of (6.21) depends only on the dependent variables Φ and ω.

Next, we will obtain an analogous equation also dependent on these two variables (Φ and ω)

Consider the thermodynamic energy equation:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \overline{V}_g \cdot \overline{\nabla}\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega &= \frac{\kappa J}{p} \qquad (6.13b)\\
\therefore \frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) + \overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega &= \frac{\kappa J}{p}\\
\therefore -\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) = \overline{V}_g \cdot \overline{\nabla}\left(\frac{\partial \Phi}{\partial p}\right) &+ \sigma\omega + \frac{\kappa J}{p}\\
\therefore \frac{\partial \chi}{\partial p} = -\overline{V}_g \cdot \overline{\nabla}\left(\frac{\partial \Phi}{\partial p}\right) &- \sigma\omega - \frac{\kappa J}{p}
\end{aligned}$$

Multiply by f_0/σ: $\dfrac{f_0}{\sigma}\dfrac{\partial \chi}{\partial p} = -\dfrac{f_0}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(\dfrac{\partial \Phi}{\partial p}\right) - f_0\omega - \dfrac{f_0 \kappa J}{\sigma p}$

Differentiate with respect to p:

$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)\right] - f_0\frac{\partial \omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right) \qquad \left[\sigma \equiv -\frac{RT_0}{p}\frac{d\ln\theta}{dp}\right] \quad (6.22)$$

The ageostrophic vertical motion, ω, has equal and opposite effects on the left-hand sides in $\left(6.21 : f_0^{-1}\nabla^2\chi\right)$ and $\left(6.22 : \dfrac{\partial}{\partial p}\left(\dfrac{f_0}{\sigma}\dfrac{\partial \chi}{\partial p}\right)\right)$

Vertical stretching $\left(\dfrac{\partial \omega}{\partial p} > 0\right)$ forces a positive tendency in the geostrophic vorticity (6.21) and a negative tendency of equal magnitude in the term on the left side in (6.22).

The left side of (6.22) can be interpreted as the local rate of change of a **normalized** static stability anomaly (i.e., a measure of the departure of static stability from S_p, its standard atmosphere value).

To demonstrate this statement:

$$\begin{aligned}
\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) &= \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)\right) = \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial p}\right)\right)\\
&= \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial t}\left(-\frac{RT}{p}\right)\right) \qquad \left\{(6.2) : \frac{\partial \Phi}{\partial p} = -\frac{RT}{p}\right\}\\
&= -f_0\frac{\partial}{\partial p}\left(\frac{R}{\sigma p}\frac{\partial T}{\partial t}\right) \qquad \left\{S_p = \frac{p\sigma}{R}\right\}\\
&= -f_0\frac{\partial}{\partial p}\left(\frac{1}{S_p}\frac{\partial T}{\partial t}\right)\\
&= -f_0\left[\frac{\partial}{\partial p}\left(\frac{1}{S_p}\right)\frac{\partial T}{\partial t} + \frac{1}{S_p}\frac{\partial}{\partial p}\frac{\partial T}{\partial t}\right]
\end{aligned}$$

Assume that S_p varies only slowly with height in the troposphere, thus S_p is nearly constant and $\frac{\partial}{\partial p}\left(S_p^{-1}\right) \approx 0$

$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) \approx -\frac{f_0}{S_p}\frac{\partial}{\partial p}\frac{\partial T}{\partial t} = -\frac{f_0}{S_p}\frac{\partial}{\partial t}\frac{\partial T}{\partial p} = -\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right)$$

From page 5 of the notes:

$$T_{tot} = T_0 + T \qquad \{T_0 : \text{basic state (standard atmosphere)}\}$$
$$\therefore T = T_{tot} - T_0$$

Therefore $\frac{\partial T}{\partial p} \sim$ local static stability anomaly

$$\therefore \frac{1}{S_p}\frac{\partial T}{\partial p} \sim \text{ Local static stability anomaly divided by the standard atmosphere static stability}$$

Take note: $\frac{f_0}{S_p}\frac{\partial T}{\partial p}$ has the same units as vorticity, and is also a **normalized** static stability value.

When the tendency of the normalized static stability anomaly > 0:

$$\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right) > 0$$
$$\therefore \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) < 0, \text{ the left side of (6.22)}$$

An air column that moves adiabatically from a region of high static stability to a region of low static stability, $\partial\omega/\partial p > 0$.

Since (6.21) and (6.22) are **analogous** equations, the relative vorticity in (6.21), $\frac{1}{f_0}\nabla^2\chi$ and the normalized static stability anomaly in (6.22) are changed by equal and opposite amounts. The normalized static stability anomaly is therefore referred to as the stretching vorticity.

Purely geostrophic motion ($\omega = 0$) is a solution to (6.21) and (6.22) only in a very special situations such as barotropic flow (no pressure dependence) or zonally symmetric flow. More general purely geostrophic flows cannot satisfy both these equations simultaneously as there are then two independent equations, and a single unknown (Φ) so that the system is overdetermined. Thus, the role of the vertical motion distribution must be to maintain consistency between the geopotential tendencies required by vorticity advection in (6.21) and thermal advection in (6.22).

Geopotential tendency

(6.21):

$$\therefore \frac{1}{f_0}\nabla^2\chi = -\overline{V}_g \cdot \overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p} \tag{1}$$

Assuming that the diabatic heating rate $J = 0$, (6.22) becomes:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right] - f_0\frac{\partial\omega}{\partial p} \tag{2}$$

(1) + (2):

$$\left[\frac{1}{f_0}\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -\overline{V}_g \cdot \overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]$$

$$\therefore \underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]}_{\text{A}}\chi = \underbrace{-f_0\overline{V}_g \cdot \overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{B}}\underbrace{-\frac{\partial}{\partial p}\left[-\frac{{f_0}^2}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{\text{C}} \tag{6.23}$$

(6.23) is often referred to as the **geopotential tendency equation**.

A. The local geopotential tendency

B. The distribution of vorticity advection

C. The thickness advection

If the distribution of Φ is known at a given time, B and C may be regarded as known forcing functions and (6.23) is a linear partial differential equation in the unknown χ.

Take note that the term A involves second derivatives in space (x, y) of the field χ, and thus generally proportional to $-\chi$.

$\chi = \dfrac{\partial\Phi}{\partial t}$ and we assume that the horizontal structure of Φ (geopotential) in the extra-tropics can be represented by a sinusoidal function:

$$\Phi = \Phi(x, y, p, t) = A(p, t)B(x, y)$$
$$\text{with } B(x, y) = \sin(kx)\cos(ly);\ k = \frac{2\pi}{L_x};\ l = \frac{2\pi}{L_y}$$
$$\therefore \chi = \frac{\partial}{\partial t}\left(A(p, t)\sin(kx)\cos(ly)\right)$$

Term A: $\left[\nabla^2 + \dfrac{\partial}{\partial p}\left(\dfrac{{f_0}^2}{\sigma}\dfrac{\partial}{\partial p}\right)\right]$ applied to χ:

$$
\begin{aligned}
\nabla^2\chi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\frac{\partial A}{\partial t}\sin(kx)\cos(ly)\right] \\
&= \frac{\partial A}{\partial t}\left(\frac{\partial^2}{\partial x^2}(\sin(kx)\cos(ly)) + \frac{\partial^2}{\partial y^2}(\sin(kx)\cos(ly))\right) \\
&= \frac{\partial A}{\partial t}\left(\cos(ly)\frac{\partial^2}{\partial x^2}(\sin(kx)) + \sin(kx)\frac{\partial^2}{\partial y^2}(\cos(ly))\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}(\sin(kx)) &= \frac{\partial}{\partial x}(k\cos(kx)) = -k^2\sin(kx) \\
\frac{\partial^2}{\partial y^2}(\cos(ly)) &= \frac{\partial}{\partial y}(-l\sin(ly)) = -l^2\cos(ly)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \nabla^2\chi &= \frac{\partial A}{\partial t}\big(\cos(ly)(-k^2\sin(kx)) + \sin(kx)(-l^2\cos(ly))\big) \\
&= \frac{\partial A}{\partial t}\sin(kx))\cos(ly)(-k^2 - l^2) \\
&= -(k^2 + l^2)\frac{\partial A}{\partial t}\sin(kx))\cos(ly) \\
&= -(k^2 + l^2)\chi \propto -\chi
\end{aligned}
$$

Since geopotential fields tend to lean **westward** with height in the mid-latitudes an upper troposphere ridge often lies over or near the surface trough:

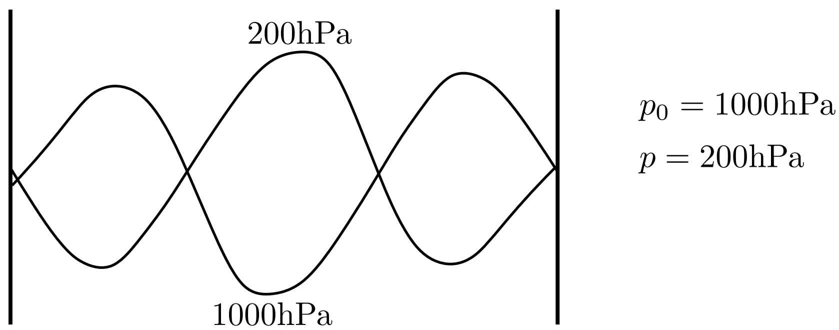


Figure 1: A full phase shift with height.

$\Phi = A(p,t)B(x,y)$; we dealt with the $B(x,y)$ part on the previous page, and so we now consider

$$
A(p,t) = Q(t)\cos\left(\frac{\pi p}{p_0}\right)
$$

$$
\begin{aligned}
\chi = \frac{\partial \Phi}{\partial t} &= \frac{\partial}{\partial t}\left(Q(t)\cos\left(\frac{\pi p}{p_0}\right)B(x,y)\right) \\
&= \cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}
\end{aligned}
$$

Regarding term A of the geopotential tendency equation, apply $\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)$ to χ:

$$
\begin{aligned}
\therefore \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\left(\cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}\right) \\
&= B\frac{\partial Q}{\partial t}\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\left(\cos\left(\frac{\pi p}{p_0}\right)\right) \\
&= B\frac{\partial Q}{\partial t}\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\left(\frac{-\pi}{p_0}\right)\sin\left(\frac{\pi p}{p_0}\right)\right)\right]
\end{aligned}
$$

Assume that the standard atmosphere static stability parameter σ, varies only slowly with height $\left(\text{i.e., } \frac{\partial}{\partial p}(\sigma^{-1}) \approx 0\right)$ in the troposphere:

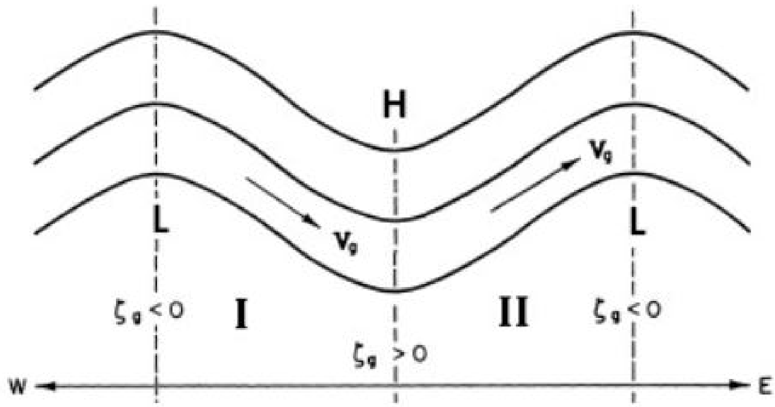
$$
\begin{aligned}
\left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= -B\frac{\partial Q}{\partial t}\frac{{f_0}^2}{\sigma}\frac{\pi}{p_0}\frac{\partial}{\partial p}\left(\sin\left(\frac{\pi p}{p_0}\right)\right) \\
\therefore \left[\frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi &= -B\frac{\partial Q}{\partial t}\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\cos\left(\frac{\pi p}{p_0}\right) \\
&= -\cos\left(\frac{\pi p}{p_0}\right)B\frac{\partial Q}{\partial t}\left(\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\right) \\
&= -\frac{{f_0}^2}{\sigma}\frac{\pi^2}{p_0^2}\chi \propto -\chi \\
&\implies \left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \propto -\chi
\end{aligned}
$$

Term A is thus generally proportional to $-\chi\left(=\frac{\partial \Phi}{\partial t}\right)$

Next, consider Term B:

$$
\begin{aligned}
-f_0\overline{V}_g \cdot \overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) &= -f_0\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right) \qquad \left[(6.15): \zeta_g = \frac{1}{f_0}\nabla^2\Phi\right] \\
&= -f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g - f_0\overline{V}_g \cdot \left(\frac{\partial f}{\partial x}\overline{i} + \frac{\partial f}{\partial y}\overline{j}\right) \\
&= -f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g - f_0(u_g\overline{i} + v_g\overline{j}) \cdot \frac{\partial f}{\partial y}\overline{j} \qquad (f \neq f(x)) \\
&= -f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g - f_0 v_g\frac{\partial f}{\partial y} \\
&= \text{geostrophic advection of relative vorticity} + \\
&\qquad \text{geostrophic advection of planetary vorticity}
\end{aligned}
$$

Consider the schematic below of a 500hPa geopotential field in the **Southern Hemisphere**:



Region I: Upstream of the $500\mathrm{hPa}$ ridge, the geostrophic wind is directed from the relative vorticity minimum at the trough towards the relative vorticity maximum at the ridge.

$$\Longrightarrow \text{ Advection of relative vorticity is positive.}$$
$$\therefore \overline{V}_g \cdot \overline{\nabla}\zeta_g > 0$$
$$\therefore f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g < 0 \qquad \text{in the SH } (f_0 < 0)$$
$$\therefore -f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g > 0 \qquad \text{in the SH}$$

At the same time $v_g < 0$ because it is directed southwards, and $\dfrac{\partial f}{\partial y} = \beta = 2\Omega\cos\phi_0/a > 0$ (both hemispheres).

$$\therefore v_g\frac{\partial f}{\partial y} < 0$$
$$\therefore f_0 v_g\frac{\partial f}{\partial y} > 0 \qquad \text{in the SH } (f_0 < 0)$$
$$\therefore -f_0 v_g\frac{\partial f}{\partial y} < 0 \qquad \text{in the SH}$$

For advection of relative vorticity:

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi \propto -\chi > 0$$
$$\therefore \chi < 0$$
$$\therefore \frac{\partial\Phi}{\partial t} < 0$$

therefore the geopotential heights are falling between the trough and the ridge axis, downstream of the trough axis.

For advection of planetary vorticity:

$$
\begin{aligned}
&-\chi < 0 \qquad \left[-f_0 v_g \frac{\partial f}{\partial y} < 0\right] \\
&\therefore \chi > 0 \\
&\therefore \frac{\partial \Phi}{\partial t} > 0
\end{aligned}
$$

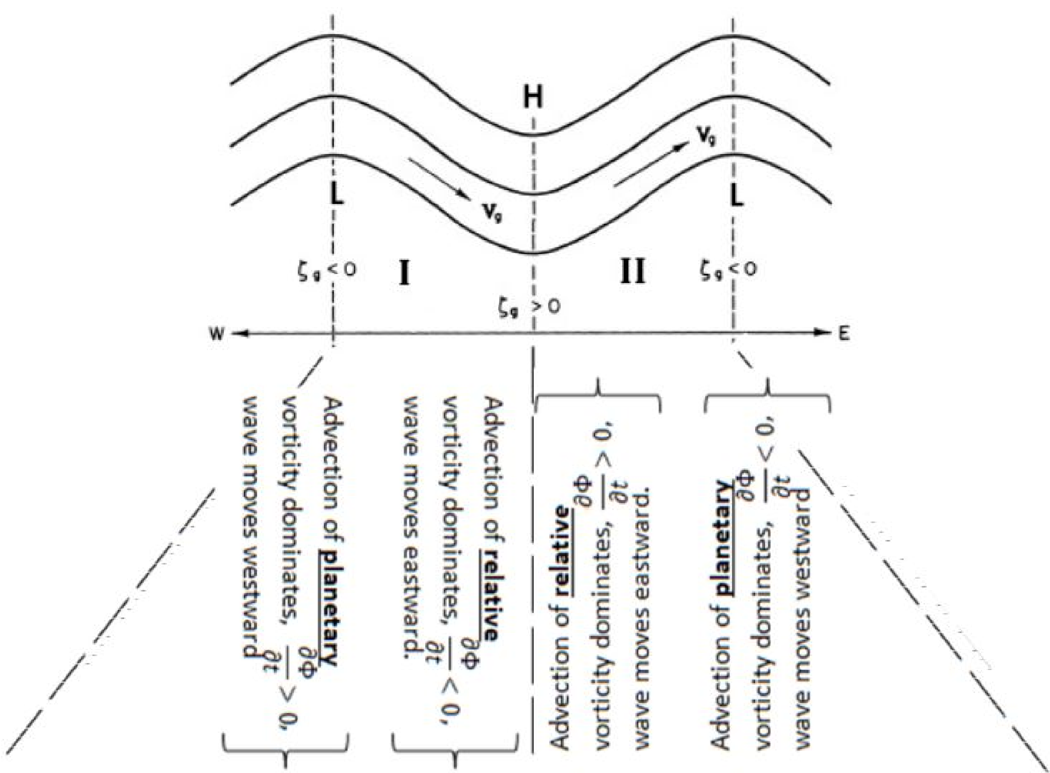
which implies that the advection of planetary vorticity results in increasing geopotential heights.

Similarly for **Region II**:

$$
\begin{aligned}
-f_0 \overline{V}_g \cdot \overline{\nabla} \zeta_g &< 0 \qquad \text{in the SH} \\
\therefore -\chi &< 0 \\
\therefore \chi &> 0 \\
\therefore \frac{\partial \Phi}{\partial t} &> 0
\end{aligned}
$$

and

$$
\begin{aligned}
-f_0 v_g \frac{\partial f}{\partial y} &> 0 \\
\therefore -\chi &> 0 \\
\therefore \chi &< 0 \\
\therefore \frac{\partial \Phi}{\partial t} &< 0
\end{aligned}
$$



For a mid-latitude disturbance of given amplitude the absolute value of the relative vorticity **increases** for **decreasing** wavelength.

Therefore for short wavelengths ($\leq 3000\text{km}$) the advection of relative vorticity tends to dominate, resulting in the disturbance moving rapidly eastwards.

For long waves ($\geq 10000\text{km}$) the planetary vorticity advection tends to dominate, resulting in these long planetary waves to be quasi-stationary.

Since $\overline{\nabla}\zeta_g$ and v_g are zero at both trough and ridge axes, the vorticity advection term is zero:

$$\begin{aligned}\text{Term B:} -f_0\overline{V}_g \cdot \overline{\nabla}\zeta_g - f_0 v_g \frac{\partial f}{\partial y} \\ &= -f_0\overline{V}_g \cdot \overline{0} - f_0(0)\frac{\partial f}{\partial y} \\ &= 0\end{aligned}$$

$\Longrightarrow$ Vorticity advection cannot change the strength of this type of disturbance at the levels where the advection is occurring, but only acts to propagate the disturbance horizontally and (as shown in the next section) to spread it vertically.

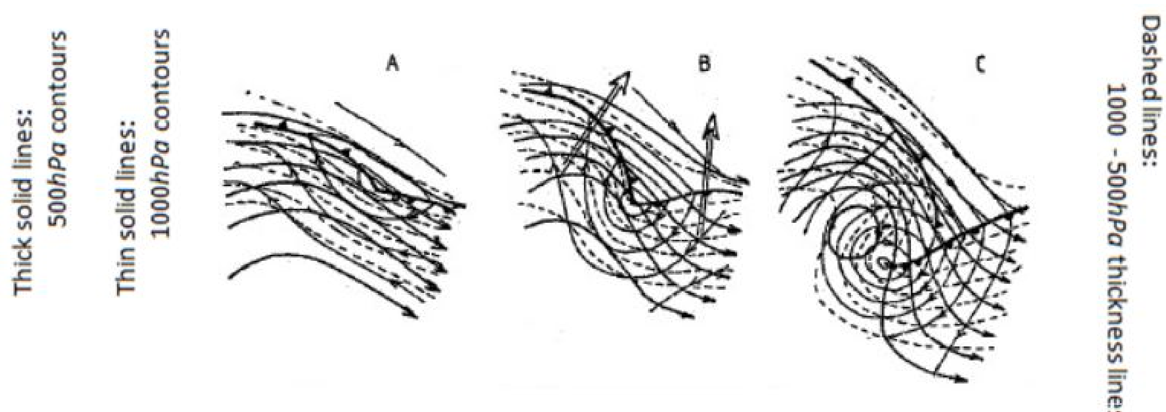
The mechanism for amplification or decay of mid-latitude synoptic systems is contained in Term C:

$$-\frac{\partial}{\partial p}\left[-\frac{{f_0}^2}{\sigma}\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] = \frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

This term is called the differential thickness advection and it tends to be a maximum at trough and ridge lines in a developing baroclinic wave.

The term $\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is proportional to the hydrostatic temperature advection, and $\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$ is proportional to the rate of change of the temperature advection with height, or the **differential temperature advection**.

Consider below an idealized schematic representation of a developing baroclinic disturbance:



In order to determine the rate of change of the temperature advection with height (or pressure) at least two levels in the vertical must be used. Here two layers are considered: 1000–500hPa layer (lower troposphere), and the 500–300hPa layer (upper troposphere).

The figure above demonstrates that a developing baroclinic disturbance is characterized by the westward tilt with height of the pressure system (the thick solid contours are to the west of the thin contours).

1. The tilting results in strong cold advection behind the cold front and strong warm advection ahead of the warm front.
2. In the upper troposphere, the tilt of the pressure system is small.

These statements are demonstrated in the figure below.

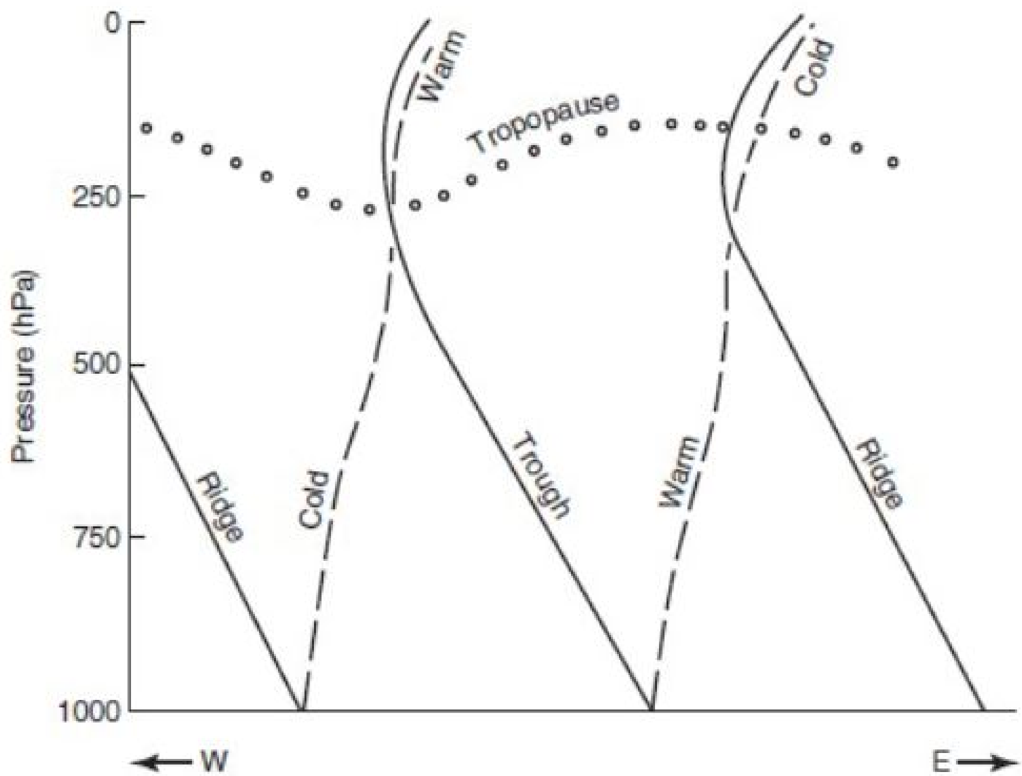


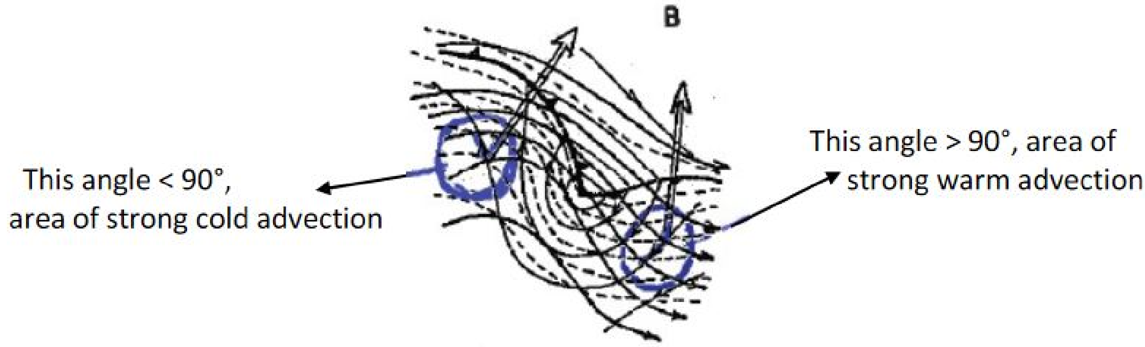
Figure 2: West-east cross section through a developing baroclinic wave. Solid lines are trough and ridge axes; dashed lines are axes of temperature extrema; the chain of open circle denotes the tropopause.

In the upper troposphere the tilt of the pressure system with height in small. The result is that the thickness pattern and the geopotential pattern become approximately parallel, which leads to thermal advection becoming small there. Term C is thus concentrated in the lower troposphere.

For the case of the lower troposphere, we want to determine the sign and the magnitude of the Term C. The horizontal thermal advection for this part of the troposphere (1000– $500\mathrm{hPa}$ layer) is given by:

$$\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$$

where $\overline{V}_g$ is the geostrophic wind at the $1000\mathrm{hPa}$ level, and $\overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is a vector that is perpendicular to the 1000– $500\mathrm{hPa}$ thickness lines. See diagram B on page 19. This vector points towards the warm sector of the low pressure system, and is shown in B for two positions (the two arrows).



The sign of the scalar product of $\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)$ is given by:

$$\left|\overline{V}_g\right| \left|\overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right| \cos\theta$$

$$\cos\theta > 0 \text{ if } \theta < 90°\text{, behind cold front}$$
$$\cos\theta < 0 \text{ if } \theta > 90°\text{, ahead of warm front}$$

Behind cold front (below $500\mathrm{hPa}$ trough):

$$\text{Thermal advection} = \overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0, \text{ cold advection}$$

Ahead of warm front (below $500\mathrm{hPa}$ ridge):

$$\text{Thermal advection} = \overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) < 0, \text{ warm advection}$$

Recalling the discussion above that for a developing system the thermal advection is much smaller in the upper troposphere than in the lower troposphere, thermal advection (both cold and warm) decreases with height. So does tropospheric pressure.

$$\therefore \frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right) > 0 \quad \text{below } 500\mathrm{hPa} \text{ trough}$$
$$\text{and} \quad \frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right) < 0 \quad \text{below } 500\mathrm{hPa} \text{ ridge}$$

Since $\frac{{f_0}^2}{\sigma}$:

$$\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] \begin{cases} < 0 & \text{at ridge} \\ > 0 & \text{at trough} \end{cases}$$

$$\begin{cases} \text{At the ridge:} & \frac{f_0{}^2}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0 \quad \text{(warm advection)} \\ & \therefore -\chi < 0 \\ & \therefore \chi > 0 \\ & \therefore \frac{\partial \Phi}{\partial t} > 0 \quad \text{(geopotential increases with time)} \end{cases} \tag{*}$$

$$\begin{cases} \text{At the trough:} & \frac{f_0{}^2}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0 \quad \text{(cold advection)} \\ & \therefore -\chi > 0 \\ & \therefore \chi < 0 \\ & \therefore \frac{\partial \Phi}{\partial t} < 0 \quad \text{(geopotential decreases with time)} \end{cases} \tag{+}$$

(*) The effect of warm advection below the $500\mathrm{hPa}$ ridge is to build the ridge.

(+) The effect of cold advection below the $500\mathrm{hPa}$ trough is to deepen the trough.

$\Longrightarrow$ The differential temperature or thickness advection intensifies the upper level troughs and ridges in a developing baroclinic system.

The advection of cold air into the air column below the $500\mathrm{hPa}$ trough will reduce the thickness of that column, and hence will lower the height of the $500\mathrm{hPa}$ surface unless there is a compensating rise in the surface pressure. Warm advection into the air column below the $500\mathrm{hPa}$ ridge will have the opposite effect.

The traditional omega equation

The vorticity equation (6.19):

$$\frac{\partial \zeta_g}{\partial t} = -\overline{V}_g \cdot \overline{\nabla}(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p}$$

ζ_g and $\overline{V}_g$ are both defined in terms of $\Phi(x, y, p, t)$:

$$\zeta_g = \frac{1}{f_0}\nabla^2 \Phi \quad \text{and} \quad \overline{V}_g = \frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi$$

Therefore the vorticity equation (6.19) can be used to diagnose ω (vertical velocity field) provided that the fields of both Φ and $\frac{\partial \Phi}{\partial t}$ are known.

- Φ : primary product of operational weather analysis
- $\frac{\partial \Phi}{\partial t}$: can only be crudely approximated from observations by taking differences over 12 hours, since upper level analyses are generally available only twice per day.

Despite this limitation, the vorticity equation method of estimating ω is usually more accurate than the continuity equation method discussed in WKD352 (the kinematic method). However, neither of these two methods of estimating ω uses the information available in the thermodynamic energy equation. Here we will develop the so-called omega equation for estimating the vertical motion by utilizing both the vorticity equation and the thermodynamic equation.

Thermodynamic energy equation (6.13b):

$$\left(\frac{\partial}{\partial t}+\overline{V}_g\cdot\overline{\nabla}\right)\left(-\frac{\partial\Phi}{\partial p}\right)-\sigma\omega=\frac{\kappa J}{p}$$

Apply the horizontal Laplacian:

$$\begin{aligned}
\nabla^2\left(\frac{\partial}{\partial t}+\overline{V}_g\cdot\overline{\nabla}\right)\left(-\frac{\partial\Phi}{\partial p}\right)-\nabla^2(\sigma\omega)&=\nabla^2\left(\frac{\kappa J}{p}\right)\\
\therefore\nabla^2\frac{\partial}{\partial t}\left(-\frac{\partial\Phi}{\partial p}\right)&=\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]+\sigma\nabla^2\omega+\frac{\kappa}{p}\nabla^2 J\\
\therefore\nabla^2\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right)&=-\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\sigma\nabla^2\omega-\frac{\kappa}{p}\nabla^2 J\\
\therefore\nabla^2\frac{\partial\chi}{\partial p}&=-\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\sigma\nabla^2\omega-\frac{\kappa}{p}\nabla^2 J
\end{aligned}\tag{6.32}$$

Rewriting the geostrophic vorticity equation:

$$\frac{1}{f_0}\nabla^2\chi=-\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)+f_0\frac{\partial\omega}{\partial p}\tag{6.21}$$

Differentiate (6.21) with respect to p:

$$\begin{aligned}
\therefore\frac{\partial}{\partial p}\left(\frac{1}{f_0}\nabla^2\chi\right)&=-\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+f_0\frac{\partial^2\omega}{\partial p^2}\\
\therefore\frac{\partial}{\partial p}\left(\nabla^2\chi\right)&=-f_0\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+{f_0}^2\frac{\partial^2\omega}{\partial p^2}
\end{aligned}\tag{6.33}$$

(6.33) – (6.32):

$$\begin{aligned}
&\left({f_0}^2\frac{\partial^2}{\partial p^2}+\sigma\nabla^2\right)\omega-f_0\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]+\frac{\kappa}{p}\nabla^2 J\\
&=\frac{\partial}{\partial p}\left(\nabla^2\chi\right)-\nabla^2\frac{\partial\chi}{\partial p}
\end{aligned}$$

Since the operators on the right hand side can be reversed:

$$\left({f_0}^2\frac{\partial^2}{\partial p^2}+\sigma\nabla^2\right)\omega=f_0\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]-\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\Phi}{\partial p}\right)\right]-\frac{\kappa}{p}\nabla^2 J$$

$$\therefore\left(\nabla^2+\frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega=\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]+\frac{1}{\sigma}\nabla^2\left[\overline{V}_g\cdot\overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]-\frac{\kappa}{\sigma p}\nabla^2 J\tag{6.34}$$

Term A : $\left(\nabla^2+\frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega$

Term B : $\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right]=\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g\cdot\overline{\nabla}\left(\zeta_g+f\right)\right]$

Term C $: \frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$

Term D $: -\frac{\kappa}{\sigma p}\nabla^2 J$, but as with the geopotential tendency equation we set $J = 0$; J is the diabatic heat rate.

The resulting omega equation:

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

The omega equation above involves only derivatives in space (not time). This equation is thus a **diagnostic equation** for the field of omega (ω) in terms of the instantaneous geopotential (Φ) field.

Remember the operator in Term A of the tendency equation? It is $\left(\nabla^2 + \frac{\partial}{\partial p}\left(\frac{{f_0}^2}{\sigma}\frac{\partial}{\partial p}\right)\right)$, and is very similar to the operator of Term A of the omega equation.

The forcing in the omega equation tends to be a maximum in the mid-troposhpere (500hPa), and ω is required to be zero at the surface and at the top of the troposphere. Therefore, for a qualitative discussion it is permissible to assume that ω has sinusoidal behaviour in both the horizontal and vertical:

$$\begin{aligned}
\omega &= W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly) \\
\therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\left(W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly)\right) \\
&= \frac{\partial^2}{\partial x^2}\sin(kx)\left[W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(ly)\right] \\
&+ \frac{\partial^2}{\partial y^2}\sin(ly)\left[W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\right] \\
&+ \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\sin\left(\frac{\pi p}{p_0}\right)\left[W_0 \sin(kx)\sin(ly)\right] \\
&= -k^2 \sin(kx)\left[W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(ly)\right] \\
&- l^2 \sin(ly)\left[W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\right] \\
&- \frac{{f_0}^2}{\sigma}\left(\frac{\pi}{p_0}\right)^2 \sin\left(\frac{\pi p}{p_0}\right)\left[W_0 \sin(kx)\sin(ly)\right] \\
&= W_0 \sin\left(\frac{\pi p}{p_0}\right)\sin(kx)\sin(ly)\left[-k^2 - l^2 - \frac{1}{\sigma}\left(\frac{\pi f_0}{p_0}\right)^2\right] \\
&= -\left[k^2 + l^2 + \frac{1}{\sigma}\left(\frac{f_0 \pi}{p_0}\right)^2\right]\omega
\end{aligned}$$

$\therefore$ Term A is proportional to $-\omega$

For synoptic-scale motions $\quad \omega = -\rho g w$

$\therefore \omega \propto -w$

$\therefore \omega < 0$ implies upward vertical motion.

Since $\omega < 0$ implies upward motion, and

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$$

$$\therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto w, \text{ the vertical velocity}$$

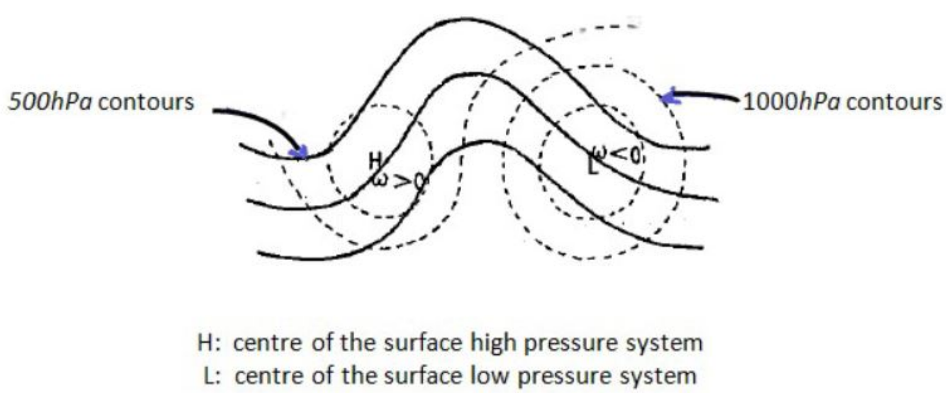
$\implies$ Upward motion is forced where the right-hand side of the omega equation is positive and downward motion is forced where it is negative.

The omega equation with negligible diabatic heating:

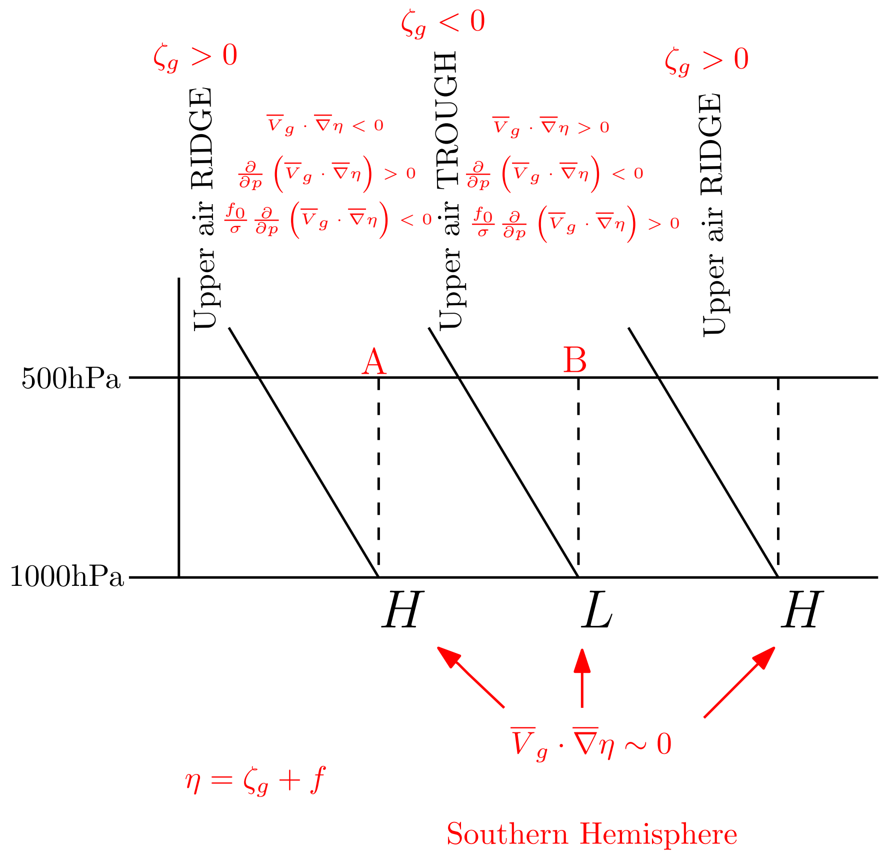
$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

Term B: $\dfrac{f_0}{\sigma}\dfrac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)\right]$, the differential vorticity advection.

This term is proportional to the rate of increase with height, or with pressure, of the advection of absolute vorticity. To discuss the role of this term we consider an idealized developing baroclinic system. Moreover, we consider a short-wave system where **relative vorticity advection is larger than the planetary vorticity advection**. The figure below shows schematically the geopotential contours at $500\mathrm{hPa}$ and $1000\mathrm{hPa}$ for such a system.



At the centres of the surface high and surface low $\overline{\nabla}\zeta_g$ and $\overline{V}_g$ must be very small: Previously it was discussed that, since $\overline{\nabla}\zeta_g$ and $\overline{V}_g$ are zero at both trough and ridge axes, the vorticity advection term is zero at the axes. However, since the H and L centres are not located exactly on the $500\mathrm{hPa}$ trough/ridge axis, $\overline{\nabla}\zeta_g$ and $\overline{V}_g$ are only very small (not zero) and so vorticity advection must be very small, that is $\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)$ must be very small.



At point A, $\overline{V}_g \cdot \overline{\nabla}\eta < 0$ since the flow is going from a ridge where $\zeta_g > 0$ towards a trough where $\zeta_g < 0$. From the 500hPa level towards the surface where the high is $\left(\overline{V}_g \cdot \overline{\nabla}\eta \sim 0\right)$, there is thus an **increase** in $\overline{V}_g \cdot \overline{\nabla}\eta$ along the vertical pressure axis. Therefore

$$\frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\eta\right) > 0$$
$$\therefore \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\eta\right) < 0 \quad \text{above the surface high}$$

At point B, $\overline{V}_g \cdot \overline{\nabla}\eta > 0$ since the flow is going from a trough where $\zeta_g < 0$ towards a ridge where $\zeta_g > 0$. From the 500hPa level towards the surface where the low is $\left(\overline{V}_g \cdot \overline{\nabla}\eta \sim 0\right)$, there is thus a **decrease** in $\overline{V}_g \cdot \overline{\nabla}\eta$ along the vertical pressure axis. Therefore

$$\frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\eta\right) < 0$$
$$\therefore \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\eta\right) > 0 \quad \text{above the surface low}$$

Considering that $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto w$, and that Term B > 0 above point L, $w > 0$, that means **ascending motion**.

Since Term B < 0 above point H, $w < 0$, that means **subsiding motion**. Now for Term C: $\frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right]$, and remembering that

$$\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right) > 0 \text{ for cold advection}$$

and

$$\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right) < 0 \text{ for warm advection}$$

Consider the diagram at the top of Page 21: East of the surface low, in the warm front zone, the warm advection tends to be a maximum and west of the surface low, behind the cold front, the cold advection tends to be a maximum.

East of surface low: Warm advection, $\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right) < 0$

However, we have shown already (twice!) that $\nabla^2 Y \propto -Y$

$$\therefore \text{ East of surface low} \quad \nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right] > 0$$

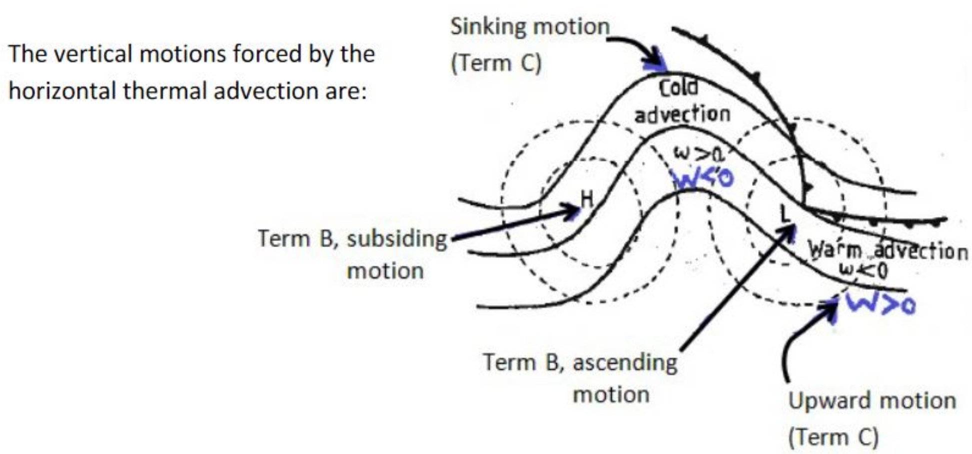
$$\therefore \frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right] > 0$$

$$w > 0 \text{ and maximum}$$

$$\text{West of surface low} \quad \nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right] < 0$$

$$\therefore \frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right] < 0$$

$$w < 0 \text{ and minimum}$$



Bonus homework: Write a short essay (not more than one page) on the so-called Dines compensation.

The Sutcliffe form of the omega equation

A problem with the traditional omega equation is that there exists significant cancellation between the two terms on the right hand side of this form of the equation. Here we are presenting an alternative approximate form of the omega equation that can be applied in synoptic analysis in the Southern Hemisphere.

First, employ the chain rule of differentiation for the two terms on the right hand side of the traditional omega equation.

The omega equation:

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)\right]}_{\text{B}} + \underbrace{\frac{1}{\sigma}\nabla^2\left[\overline{V}_g \cdot \overline{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{C}}$$

Apply the chain rule of differentiation on Term B:

$$\begin{aligned} &\frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p}\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + \overline{V}_g \cdot\left(\frac{1}{f_0}\frac{\partial \nabla^2 \Phi}{\partial p} + \frac{\partial f}{\partial p}\right)\right] \\ &= \frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p}\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + \frac{1}{\sigma}\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial \nabla^2\Phi}{\partial p}\right) \qquad (6.35a) \end{aligned}$$

Also for Term C:

$$\begin{aligned} &\frac{1}{\sigma}\left(\nabla^2\overline{V}_g\right)\cdot\left(-\frac{\partial\Phi}{\partial p}\right) + \frac{1}{\sigma}\overline{V}_g\cdot\nabla^2\left(\overline{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right) \\ &= -\frac{1}{\sigma}\left(\left(\nabla^2 u_g\right)\bar{i} + \left(\nabla^2 v_g\right)\bar{j}\right)\cdot\left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j}\right)\left(\frac{\partial\Phi}{\partial p}\right) - \frac{1}{\sigma}\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\nabla^2\Phi}{\partial p}\right) \\ &= -\frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial p}\right)\right] - \frac{1}{\sigma}\overline{V}_g\cdot\overline{\nabla}\left(\frac{\partial\nabla^2\Phi}{\partial p}\right) \qquad (6.35b) \end{aligned}$$

<u>NOTE</u>: The last terms in (6.35a) and (6.35b) are equal and opposite, therefore they cancel.

$$\begin{aligned} \therefore \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega &= \text{Term B} + \text{Term C} \\ &= \frac{f_0}{\sigma}\left[\frac{\partial\overline{V}_g}{\partial p}\cdot\overline{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] \qquad (B1) \\ &- \frac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial p}\right)\right] \qquad (C1) \end{aligned}$$

Scale analysis of these two expanded terms can help to compare the relative sizes of the two terms in order to reduce them.

$$\underline{\textbf{Note}}: \qquad R = 287\,\mathrm{J\,K^{-1}\,kg^{-1}}$$
$$1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$$
$$1\,\mathrm{J} = 1\,\mathrm{N\,m}$$

Term B1: $\dfrac{f_0}{\sigma}\left[\dfrac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\dfrac{1}{f_0}\nabla^2\Phi + f\right)\right] = \dfrac{f_0}{\sigma}\left[\dfrac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right]$

$$\begin{aligned}\frac{\partial \overline{V}_g}{\partial p} \sim \frac{10\,\mathrm{m\,s^{-1}}}{10 \times 10^2\,\mathrm{Pa}} &= \frac{1\,\mathrm{m\,s^{-1}}}{10^2\,\mathrm{N\,m^{-2}}} \\ &= 10^{-2}\,\mathrm{N^{-1}\,m^3\,s^{-1}} \\ &= 10^{-2}\,\mathrm{kg^{-1}\,m^{-1}\,s^2\,m^3\,s^{-1}} \\ &= 10^{-2}\,\mathrm{kg^{-1}\,m^2\,s}\end{aligned}$$

$$\begin{aligned}\overline{\nabla}\left(\zeta_g + f\right) &\sim \frac{1}{L}\left(10^{-5} - 10^{-4}\right)\mathrm{s^{-1}} \\ &\sim 10^{-6}\,\mathrm{m^{-1}}\left(10^{-4}\right)\mathrm{s^{-1}} \\ &= 10^{-10}\,\mathrm{m^{-1}\,s^{-1}}\end{aligned}$$

$$\begin{aligned}\therefore B1 &\sim \frac{10^{-4}\,\mathrm{s^{-1}}}{\sigma}\left[10^{-2}\,\mathrm{kg^{-1}\,m^2\,s}\right]\left[10^{-10}\,\mathrm{m^{-1}\,s^{-1}}\right] \\ &= \frac{1}{\sigma}10^{-16}\,\mathrm{kg^{-1}\,m\,s^{-1}}\end{aligned}$$

Term C1: $-\dfrac{1}{\sigma}\left[\left(\nabla^2 u_g\right)\dfrac{\partial}{\partial x}\left(\dfrac{\partial \Phi}{\partial p}\right) + \left(\nabla^2 v_g\right)\dfrac{\partial}{\partial y}\left(\dfrac{\partial \Phi}{\partial p}\right)\right]$

$$\begin{aligned}\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} &\sim \frac{10^2\,\mathrm{J\,K^{-1}\,kg^{-1}}\left(10^2\,\mathrm{K}\right)}{1000 \times 10^2\,\mathrm{Pa}} \\ &= \frac{10^4\,\mathrm{N\,m\,kg^{-1}}}{10^5\,\mathrm{N\,m^{-2}}} = 10^{-1}\,\mathrm{m^3\,kg^{-1}}\end{aligned}$$

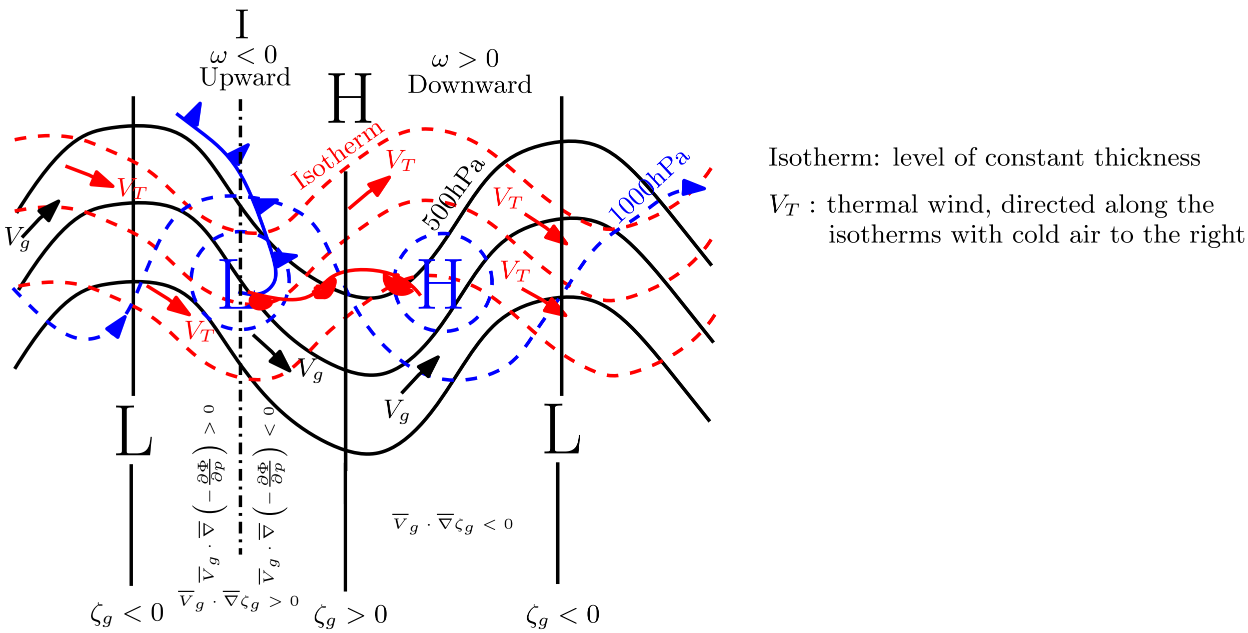
$$\begin{aligned}\therefore C1 \sim \frac{1}{\sigma}\left(\frac{1}{L^2}U\frac{1}{L}\left(10^{-1}\,\mathrm{m^3\,kg^{-1}}\right)\right) &\sim \frac{1}{\sigma}\left(10^6\,\mathrm{m}\right)^{-3} 10\,\mathrm{m\,s^{-1}}\,10^{-1}\,\mathrm{m^3\,kg^{-1}} \\ &= \frac{1}{\sigma}10^{-18}\,\mathrm{kg^{-1}\,m\,s^{-1}}\end{aligned}$$

$$\therefore B1 \sim 100 \times C1$$

$$\Longrightarrow \left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \approx \frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right]$$

The remaining term on the right of this equation represents the advection of absolute vorticity by the thermal wind. The left hand side, $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega$, is proportional to $-\omega$. When $\omega < 0$ upward vertical motion is implied, and the left hand side is proportional to the vertical velocity. Therefore, upward motion is forced where $\frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right] > 0$, and downward motion is forced where $\frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right] < 0$.

Consider an idealized schematic for a developing synoptic-scale system in the Southern Hemisphere mid-latitudes.



Take note that the 500 hPa contours lead the 1000 hPa contours due to the westward tilt of the system. The result is that the 500 hPa geopotential field lead the isotherm pattern on the figure. The thermal wind, V_T, is parallel to the isotherms, and so the term on the right that represents the advection of absolute vorticity by the thermal wind can be estimated from the change of absolute vorticity along the isotherms.

Keep in mind that we are working here with short-wavelength synoptic-scale systems where relative vorticity advection dominates planetary vorticity advection. Consider the region marked I on the figure of the idealized system. In that region a surface low pressure system is located, and above this surface low at the 500 hPa level the relative vorticity advection is a positive maximum since $\left(\overline{V}_g \cdot \overline{\nabla}\zeta_g\right)_{500\,\mathrm{hPa}} > 0$. This positive advection term is over the surface low pressure center and subsequently contributes to spin-up of the cyclone because the wind is blowing higher positive vorticity into the area of the surface low. However, on the vertical axis of this surface low pressure $\left(\overline{V}_g \cdot \overline{\nabla}\zeta_g\right)_{1000\,\mathrm{hPa}} \approx 0$ because $\left(\overline{\nabla}\zeta_g\right)_{1000\,\mathrm{hPa}} \approx 0$.

Apply the operator $\dfrac{\partial}{\partial p}$ to the absolute vorticity advection term. We get $\dfrac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\left(\zeta_g + f\right)\right) \approx \dfrac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\zeta_g\right)$ for the short-wavelength system considered here.

$$
\begin{aligned}
\frac{\partial}{\partial p}\left(\overline{V}_g \cdot \overline{\nabla}\zeta_g\right) &= \frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\zeta_g + \overline{V}_g \cdot \frac{\partial}{\partial p}\left(\overline{\nabla}\zeta_g\right) \\
&= \frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\zeta_g \quad \text{since } \ \zeta_g = \zeta_g(x, y)
\end{aligned}
$$

We can write

$$
\begin{aligned}
\frac{\delta \overline{V}_g}{\delta p} \cdot \overline{\nabla}\zeta_g &= \frac{\delta(\overline{V}_g \cdot \overline{\nabla}\zeta_g)}{\delta p} \\
&= \frac{(\overline{V}_g \cdot \overline{\nabla}\zeta_g)_{1000\,\mathrm{hPa}} - (\overline{V}_g \cdot \overline{\nabla}\zeta_g)_{500\,\mathrm{hPa}}}{1000\,\mathrm{hPa} - 500\,\mathrm{hPa}} \\
&\approx \frac{0 - \text{positive value}}{\text{positive value}} < 0 \\
&\implies \frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\zeta_g < 0
\end{aligned}
$$

For short-wave systems:

$$
\begin{aligned}
&\frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right) < 0 \\
&\therefore \frac{f_0}{\sigma}\left[\frac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right] > 0
\end{aligned}
$$

In region I where $\dfrac{f_0}{\sigma}\left[\dfrac{\partial \overline{V}_g}{\partial p} \cdot \overline{\nabla}\left(\zeta_g + f\right)\right]$ has now been demonstrated to be a positive value, upward motion is forced. Using similar arguments for the region over the surface high pressure system, downward motion is forced. Therefore, upward (downward) motion is forced east (west) of the $500\,\mathrm{hPa}$ trough above the surface low (high) pressure system.

Revisiting the idealized schematic for a developing system, upward motion occurs where relative vorticity increases moving left to right along an isotherm, and downward motion occurs where relative vorticity decreases moving left to right along an isotherm. Notwithstanding the increase in relative vorticity when moving along the isotherm, in the Southern Hemisphere cyclonic storms are associated with negative relative vorticity. Moreover, since $\zeta_g \propto \Phi$ in the Southern Hemisphere the negative vorticity is associated with negative geopotential deviations in region I, which results in the $1000 - 500\,\mathrm{hPa}$ thickness decreasing there leading to a developing trough.

Cold advection occurs behind the cold front, i.e. $\overline{V}_g \cdot \overline{\nabla}\left(-\dfrac{\partial \Phi}{\partial p}\right) > 0$, and warm advection ahead of the warm front, i.e. $\overline{V}_g \cdot \overline{\nabla}\left(-\dfrac{\partial \Phi}{\partial p}\right) < 0$. As a result, the horizontal temperature advection is small above the centre of the surface low in region I. Therefore, in order to cool the atmosphere—as required by the thickness tendency—is by adiabatic (no heat or mass exchange with the environment) cooling through the vertical motion field. As a result, in the presence of differential vorticity advection, the vertical motion maintains a field in which temperature and thickness are proportional (remember that from the thermal wind equation we have seen that $\Phi_1 - \Phi_0 = R\langle T\rangle \ln\left(\frac{p_0}{p_1}\right)$). Because of this proportionality, the vertical motion maintains the temperature field, which is determined by the geopotential field.

The Q-vector

Objective: To better appreciate the essential role of the divergent **a**geostrophic motion in quasi-geostrophic flow.

Here we examine separately the rates of change, following the geostrophic wind, of the vertical shear of the geostrophic wind and of the horizontal temperature gradient.

The approximate horizontal momentum equation:

$$\frac{D_g\overline{V}_g}{Dt} = -f_0\overline{k}\times\overline{V}_a - \beta y\overline{k}\times\overline{V}_g \tag{6.11}$$

Quasi-geostrophic momentum equations:

$$(6.16):\ \frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0 \tag{6.38}$$

$$(6.17):\ \frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0 \tag{6.39}$$

Quasi-geostrophic thermodynamic energy equation

$$\left(\frac{\partial}{\partial t} + \overline{V}_g\cdot\overline{\nabla}\right)T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p} \tag{6.13a}$$

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \overline{V}_g\cdot\overline{\nabla}$$

$$\therefore \frac{D_g T}{Dt} - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p} \tag{6.40}$$

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$\text{and }\ p\frac{\partial u_g}{\partial p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p \tag{3.29}$$

On mid-latitude β-plane:

$$f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \quad \text{and} \quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

Vector form:

$$f_0\overline{k}\times\frac{\partial\overline{V}_g}{\partial p} = \frac{R}{p}\overline{\nabla}T$$

(**Bonus homework**)

Obtaining equation for the evolution of the thermal wind components:

$$
\begin{aligned}
&f_0\frac{\partial}{\partial p}((\mathbf{6.38})) = f_0\frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g\right) = 0\\
&\therefore f_0\frac{\partial}{\partial p}\left[\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)u_g\right] - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = 0\\
&\therefore f_0\left[\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + \frac{\partial}{\partial p}\left(u_g\frac{\partial u_g}{\partial x}\right) + \frac{\partial}{\partial p}\left(v_g\frac{\partial u_g}{\partial y}\right)\right] - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = 0\\
&\therefore f_0\frac{\partial^2 u_g}{\partial p\partial t} + f_0\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + f_0 u_g\frac{\partial^2 u_g}{\partial p\partial x} + f_0\frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} + f_0 v_g\frac{\partial^2 u_g}{\partial p\partial y} - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = 0\\
&\therefore f_0\frac{\partial^2 u_g}{\partial p\partial t} + f_0 u_g\frac{\partial^2 u_g}{\partial p\partial x} + f_0 v_g\frac{\partial^2 u_g}{\partial p\partial y} + f_0\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + f_0\frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = 0\\
&\therefore f_0\left[\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right]\frac{\partial u_g}{\partial p} + f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = 0
\end{aligned}
$$

$$
\Longrightarrow \frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \tag{6.43a}
$$

Similarly:

$$
\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \tag{6.43b}
$$

Bonus homework: Derive (6.43b)

Reminder:

$$
(3.29):\quad f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \implies -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]
$$

$$
(3.28):\quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \implies -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right]
$$

However, the divergence of the geostrophic wind vanishes: $\overline{\nabla}\cdot\overline{V}_g = 0$

$$
\therefore \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \implies \frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y}
$$

$$
\begin{aligned}
\therefore -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] &= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\left(-\frac{\partial v_g}{\partial y}\right) - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]\\
&= \frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]\\
&= -Q_2
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &\equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right]\\
&= -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\bar{i}+\frac{\partial v_g}{\partial y}\bar{j}\right)\cdot\left(\frac{\partial T}{\partial x}\bar{i}+\frac{\partial T}{\partial y}\bar{j}\right)\\
&= -\frac{R}{p}\frac{\partial}{\partial y}\left(u_g\bar{i}+v_g\bar{j}\right)\cdot\left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}\right)T\\
&= -\frac{R}{p}\frac{\partial}{\partial y}\overline{V}_g\cdot\overline{\nabla}T
\end{aligned}
\tag{6.45a}
$$

For

$$
\begin{aligned}
\therefore -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right] &= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x}+\frac{\partial T}{\partial x}\left(\frac{\partial u_g}{\partial x}\right)\right]\\
&= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right]\\
&= Q_1
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &\equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right]\\
&= -\frac{R}{p}\frac{\partial}{\partial x}\left(u_g\bar{i}+v_g\bar{j}\right)\cdot\left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}\right)T\\
&= -\frac{R}{p}\frac{\partial}{\partial x}\overline{V}_g\cdot\overline{\nabla}T
\end{aligned}
\tag{6.45b}
$$

Consider the thermodynamic energy equation

$$
\frac{D_gT}{Dt}-\frac{\sigma p}{R}\omega=\frac{J}{c_p}
\tag{6.40}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x}((\mathbf{6.40})):\ \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right]T=\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x}\\
&\therefore\frac{\partial^2T}{\partial x\partial t}+\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+u_g\frac{\partial^2T}{\partial x^2}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}+v_g\frac{\partial^2T}{\partial x\partial y}=\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x}\\
&\therefore\frac{\partial^2T}{\partial x\partial t}+u_g\frac{\partial^2T}{\partial x^2}++v_g\frac{\partial^2T}{\partial x\partial y}=-\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x}\\
&\therefore\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\frac{\partial T}{\partial x}=-\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\frac{\sigma p}{R}\frac{\partial\omega}{\partial x}+\frac{1}{c_p}\frac{\partial J}{\partial x}
\end{aligned}
$$

Multiply throughout by $\dfrac{R}{p}$:

$$
\begin{aligned}
\therefore\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) &= -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial x}+\frac{R}{pc_p}\frac{\partial J}{\partial x}\\
&= -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial x}+\frac{\kappa}{p}\frac{\partial J}{\partial x}\qquad\left[\kappa\equiv\frac{R}{c_p}\right]
\end{aligned}
\tag{6.46a}
$$

$$
\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right)=-\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)+\sigma\frac{\partial\omega}{\partial y}+\frac{\kappa}{p}\frac{\partial J}{\partial y}
\tag{6.46b}
$$

Bonus homework: Derive (6.46b). Hint: Start by $\frac{\partial}{\partial y}$(**(6.40)**)

Revisiting (6.43a), and remembering

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y}, \quad f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \quad \text{and} \quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x}$$

$$\begin{aligned}
\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) &= -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\left(-\frac{\partial v_g}{\partial y}\right) - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= \frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right] + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \\
&= -Q_2 + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \qquad (6.47)
\end{aligned}$$

and for (6.46b):

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} = Q_2 + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \qquad (6.48)$$

$$\begin{aligned}
\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) &= -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \qquad (6.43b) \\
&= -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \\
&= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \\
&= Q_1 - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \qquad (6.49)
\end{aligned}$$

and for (6.46a)

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} = Q_1 + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} \qquad (6.50)$$

We set out to examine separately the rates of change $\left(\frac{D_g}{Dt}\right)$ of the vertical shear $\left(\frac{\partial}{\partial p}\overline{V}_g\right)$ of the geostrophic wind and of the horizontal temperature gradient $(\overline{\nabla}T)$. We subsequently derived two sets of equations describing these relationships. The first set of equations, in Q_2, is:

$$(6.47): \quad \frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -Q_2 + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \qquad \text{(shear)}$$

$$(6.48): \quad \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \qquad \text{(temperature gradient)}$$

(6.48) − (6.47):

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) - Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} - \left[\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) + Q_2 - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p}\right] = 0$$

$$\therefore \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y} - f_0\frac{\partial u_g}{\partial p}\right) - 2Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} = 0$$

Take note that

$$f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \tag{6.41a}$$

$$\therefore \frac{R}{p}\frac{\partial T}{\partial y} - f_0\frac{\partial u_g}{\partial p} = 0$$

$$\therefore -2Q_2 - \sigma\frac{\partial \omega}{\partial y} - \frac{\kappa}{p}\frac{\partial J}{\partial y} + {f_0}^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} = 0$$

$$\sigma\frac{\partial \omega}{\partial y} - {f_0}^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = -2Q_2 - \frac{\kappa}{p}\frac{\partial J}{\partial y} \tag{6.51}$$

(6.50) + (6.49):

$$\therefore \frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p} + \frac{R}{p}\frac{\partial T}{\partial x}\right) - Q_1 + {f_0}^2\frac{\partial u_a}{\partial p} + f_0\beta y\frac{\partial u_g}{\partial p} - Q_1 - \sigma\frac{\partial \omega}{\partial x} - \frac{\kappa}{p}\frac{\partial J}{\partial x} = 0$$

Take note that

$$f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \tag{6.41b}$$

$$\therefore f_0\frac{\partial v_g}{\partial p} + \frac{R}{p}\frac{\partial T}{\partial x} = 0$$

$$\therefore -2Q_1 - \frac{\kappa}{p}\frac{\partial J}{\partial x} = -{f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} + \sigma\frac{\partial \omega}{\partial x}$$

$$\sigma\frac{\partial \omega}{\partial x} - {f_0}^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} = -2Q_1 - \frac{\kappa}{p}\frac{\partial J}{\partial x} \tag{6.52}$$

$\frac{\partial}{\partial x}$((**6.52**)):

$$\sigma\frac{\partial^2 \omega}{\partial x^2} - {f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right) - f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right) = -2\frac{\partial Q_1}{\partial x} - \frac{\kappa}{p}\frac{\partial^2 J}{\partial x^2} \tag{A}$$

$\frac{\partial}{\partial y}$((**6.51**)):

$$\sigma\frac{\partial^2 \omega}{\partial y^2} - {f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right) - f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right) - f_0\beta\frac{\partial v_g}{\partial p} = -2\frac{\partial Q_2}{\partial y} - \frac{\kappa}{p}\frac{\partial^2 J}{\partial y^2} \tag{B}$$

(A)+(B):

$$\sigma\frac{\partial^2\omega}{\partial x^2}+\sigma\frac{\partial^2\omega}{\partial y^2}-{f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right)-{f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right)-f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right)$$
$$-f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right)-f_0\beta\frac{\partial v_g}{\partial p}=-2\frac{\partial Q_1}{\partial x}-2\frac{\partial Q_2}{\partial y}-\frac{\kappa}{p}\frac{\partial^2 J}{\partial x^2}-\frac{\kappa}{p}\frac{\partial^2 J}{\partial y^2}$$
$$\therefore\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial p}\right)-{f_0}^2\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial p}\right)-f_0\beta y\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right)$$
$$-f_0\beta y\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial p}\right)-f_0\beta\frac{\partial v_g}{\partial p}=-2\overline{\nabla}\cdot\overline{Q}-\frac{\kappa}{p}\nabla^2 J$$

We have:

1) The divergence of the geostrophic wind vanishes:

$$\overline{\nabla}\cdot\overline{V}_g=0$$
$$\therefore\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}=0$$
$$\therefore\frac{\partial u_g}{\partial x}=-\frac{\partial v_g}{\partial y}$$

2) (6.12): $\dfrac{\partial u_a}{\partial x}+\dfrac{\partial v_a}{\partial y}=-\dfrac{\partial\omega}{\partial p}$

$$\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial p}\left(\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}\right)-f_0\beta y\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}\right)-f_0\beta\frac{\partial v_g}{\partial p}$$
$$=-2\overline{\nabla}\cdot\overline{Q}-\frac{\kappa}{p}\nabla^2 J$$
$$\therefore\sigma\nabla^2\omega-{f_0}^2\frac{\partial}{\partial p}\left(-\frac{\partial\omega}{\partial p}\right)=-2\overline{\nabla}\cdot\overline{Q}+f_0\beta\frac{\partial v_g}{\partial p}-\frac{\kappa}{p}\nabla^2 J$$
$$\therefore\sigma\nabla^2\omega+{f_0}^2\frac{\partial^2\omega}{\partial p^2}=-2\overline{\nabla}\cdot\overline{Q}+f_0\beta\frac{\partial v_g}{\partial p}-\frac{\kappa}{p}\nabla^2 J\qquad(6.53)$$

$\Longrightarrow$ the Q-vector form of the omega equation.

From (6.45a,b):

$$\overline{Q}=(Q_1,Q_2)$$
$$=\left(-\frac{R}{p}\frac{\partial}{\partial x}\overline{V}_g\cdot\overline{\nabla}T,-\frac{R}{p}\frac{\partial}{\partial y}\overline{V}_g\cdot\overline{\nabla}T\right)\qquad(6.54)$$

Outside regions of active precipitation, diabatic heating is due primarily to net radiative heating, which is weak in the troposphere. Therefore, the Laplacian of the diabatic heating can be neglected. Also, the term related to the beta (β) effect is generally small for synoptic scale motion, and is subsequently also neglected. The resulting Q vector form of the omega equation is

$$\sigma\nabla^2\omega+{f_0}^2\frac{\partial^2\omega}{\partial p^2}=-2\overline{\nabla}\cdot\overline{Q}$$

Next we will discuss $\overline{Q}$ as a forcing function of the omega equation.

Previously it was demonstrated that $\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$

Multiplying throughout with σ $(\sigma > 0)$: $\left(\sigma\nabla^2 + {f_0}^2\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$

So when $\left({f_0}^2\frac{\partial^2}{\partial p^2} + \sigma\nabla^2\right)\omega > 0$, we have ascending (upward) motion $(\omega < 0)$

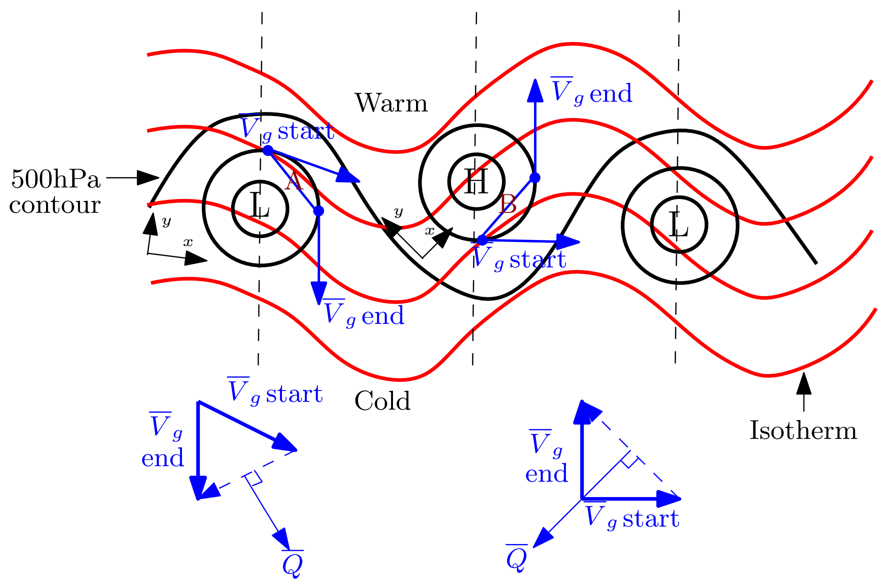
$$\therefore -2\overline{\nabla}\cdot\overline{Q} > 0$$
$$\therefore \overline{\nabla}\cdot\overline{Q} < 0$$

$\therefore$ Negative divergence of $\overline{Q}$, i.e. convergence of $\overline{Q}$, leads to ascending motion.

Similarly, when $\left({f_0}^2\frac{\partial^2}{\partial p^2} + \sigma\nabla^2\right)\omega < 0$, we have descending (downward) motion $(\omega > 0)$

$$\therefore -2\overline{\nabla}\cdot\overline{Q} < 0$$
$$\therefore \overline{\nabla}\cdot\overline{Q} > 0$$

$\therefore$ Divergence of $\overline{Q}$ leads to descending motion.



Consider an idealized developing synoptic-scale system in the Southern Hemisphere mid-latitudes at the $500\mathrm{hPa}$ level. The Q-vector direction and magnitude can be estimated by referring the motion to a Cartesian coordinate system. In this coordinate system, the x-axis is parallel to the local isotherm and the y-axis is perpendicular to the isotherm. Since warm air is to the left of an observer moving along an isotherm, temperature increases in the positive y-direction in the Southern Hemisphere. In this configuration, the

Q-vector may be simplified:

$$
\begin{aligned}
Q_1 &= -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} \quad \text{since } \frac{\partial T}{\partial x} = 0 \ (x\text{-axis parallel to isotherm}) \\
Q_2 &= -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] \\
&= -\frac{R}{p}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} \\
&= \frac{R}{p}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y} \quad \text{since } \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \\
\implies \overline{Q} &= -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\overline{i} - \frac{\partial u_g}{\partial x}\overline{j}\right) \\
\text{consider } -\overline{k} \times \left(\frac{\partial u_g}{\partial x}\overline{i} + \frac{\partial v_g}{\partial x}\overline{j}\right) &= -\overline{k} \times \frac{\partial \overline{V}_g}{\partial x} \\
&= -\frac{\partial u_g}{\partial x}\overline{j} + \frac{\partial v_g}{\partial x}\overline{i} \\
\therefore \overline{Q} &= -\frac{R}{p}\frac{\partial T}{\partial y}\left(-\overline{k} \times \frac{\partial \overline{V}_g}{\partial x}\right) \\
&= \frac{R}{p}\frac{\partial T}{\partial y}\left(\overline{k} \times \frac{\partial \overline{V}_g}{\partial x}\right)
\end{aligned}
$$

The Q-vector can be obtained by considering the vectorial change of $\overline{V}_g$ along the isotherm. Consider two cases, A and B, of an observer moving along an isotherm. For each case, draw an arrow describing the geostrophic wind vector observed at the start of each movement. Draw a second arrow showing the geostrophic wind vector at the end of each movement. Next, draw the vector difference from the head of the start vector to the head of the end vector. The Q-vector direction points $90°$ to the left (anti-clockwise) from the geostrophic difference vector in the Southern Hemisphere as dictated by the reduced Q-vector equation above. The resulting vector, multiplied by $\partial T/\partial y$, provides its magnitude.

Near the $500\mathrm{hPa}$ low, the geostrophic wind change vector produces a Q-vector parallel to the thermal wind, while near the high the Q-vector is anti-parallel to the thermal wind. The two Q-vectors thus converge in the area between the trough and ridge lines where we have already shown upward motion to occur.

Ageostrophic flow

The characteristic horizontal scale of the geostrophic wind in the mid-latitude troposphere is about 10 to $20\,\mathrm{m\,s^{-1}}$, while the scale of the ageostrophic wind is an order of magnitude smaller, often only $1-2\,\mathrm{m\,s^{-1}}$. Although the ageostrophic flow is only a small component of the wind field, the upward motion, omega (ω), is determined only by its ageostrophic part. Here we will further demonstrate the significance of the ageostrophic wind components.

Consider the following thermal wind relationship

$$f_0 \frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}$$

Then the evolution of the thermal wind components leads to

$$\frac{D_g}{Dt}\left(f_0 \frac{\partial u_g}{\partial p}\right) = -Q_2 + {f_0}^2 \frac{\partial v_a}{\partial p} + f_0 \beta y \frac{\partial v_g}{\partial p} \tag{1}$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma \frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \tag{2}$$

Assume that diabatic heating is small enough to disregard and consider the flow to be purely geostrophic, i.e. $\overline{V}_a = \overline{0}$, and $\omega = 0$ because ω is determined only by the ageostrophic part of the wind field:

$$-\frac{\partial \omega}{\partial p} = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

Equations (1) and (2) are reduced to

$$\frac{D_g}{Dt}\left(f_0 \frac{\partial u_g}{\partial p}\right) = -Q_2 + f_0 \beta y \frac{\partial v_g}{\partial p} \tag{3}$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 \tag{4}$$

Scale analysis is subsequently performed in order to estimate the magnitudes of the various terms of equation (3). First, consider the left-hand side of equation (3)

$$\frac{D_g}{Dt}\left(f_0 \frac{\partial u_g}{\partial p}\right) = \frac{\partial}{\partial t}\left(f_0 \frac{\partial u_g}{\partial p}\right) + u_g \frac{\partial}{\partial x}\left(f_0 \frac{\partial u_g}{\partial p}\right) + v_g \frac{\partial}{\partial y}\left(f_0 \frac{\partial u_g}{\partial p}\right)$$

$$\frac{\partial}{\partial t}\left(f_0 \frac{\partial u_g}{\partial p}\right) \sim \frac{1}{L/U} f_0 \frac{U}{\delta p} = \frac{f_0 U^2}{L \delta p}$$

$$u_g \frac{\partial}{\partial x}\left(f_0 \frac{\partial u_g}{\partial p}\right) \text{ and } v_g \frac{\partial}{\partial y}\left(f_0 \frac{\partial u_g}{\partial p}\right) \sim U \frac{1}{L} f_0 \frac{U}{\delta p} = \frac{f_0 U^2}{L \delta p}$$

Consider the right-hand side of equation (3)

$Q_2 = -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right]$, but consider (3.28) and (3.29)

$$p \frac{\partial v_g}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x} \implies \frac{\partial T}{\partial x} = -\frac{fp}{R}\frac{\partial v_g}{\partial p},$$

$$p \frac{\partial u_g}{\partial p} = \frac{R}{f}\frac{\partial T}{\partial y} \implies \frac{\partial T}{\partial y} = \frac{fp}{R}\frac{\partial u_g}{\partial p}$$

$$
\begin{aligned}
\therefore Q_2 &= -\frac{R}{p}\frac{fp}{R}\left[-\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} + \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right] \\
&= f\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right] \\
&= (f_0 + \beta y)\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right]
\end{aligned}
$$

Assuming that the meridional displacement, y, is at most equal to the length scale of $10^6\,\mathrm{m}$, then

$$
\begin{aligned}
\beta y &\sim 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1} \times (10^6\,\mathrm{m}) \\
&= 10^{-5}\,\mathrm{s}^{-1} \ll f_0
\end{aligned}
$$

$$
\begin{aligned}
\therefore Q_2 &= f_0\left[\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial p} - \frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial p}\right] \\
&\sim \frac{f_0 U^2}{L\delta p}
\end{aligned}
$$

Scale analysis has therefore shown that the order of magnitude of the left-hand side of the equation and Q_2 is the same, which is

$$
\begin{aligned}
\frac{f_0 U^2}{L\delta p} &\sim \frac{10^{-4}\,\mathrm{s}^{-1} \times (10\,\mathrm{m}\,\mathrm{s}^{-1})^2}{(10^6\,\mathrm{m}) \times 1000\,\mathrm{kg}\,\mathrm{m}^{-1}\,\mathrm{s}^{-2}} \\
&= 10^{-11}\,\mathrm{kg}^{-1}\,\mathrm{m}^2\,\mathrm{s}^{-1}
\end{aligned}
$$

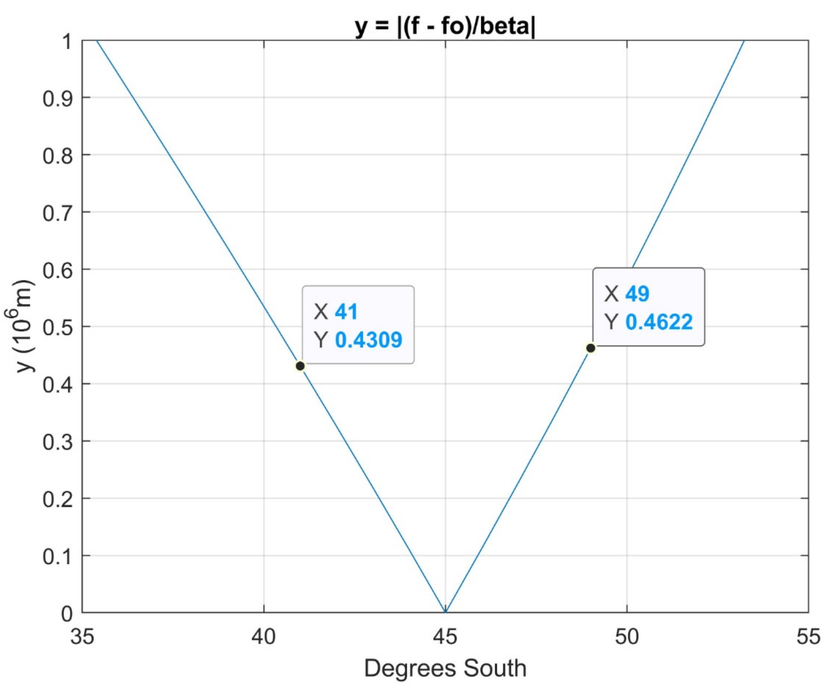


Figure 3: Meridional displacement in absolute terms in the Southern Hemisphere.

Since the meridional displacement $y = \beta^{-1}(f - f_0)$, the figure above shows that meridional displacements between about $41°$S and $49°$S, y can be approximated by $10^5\,\mathrm{m}$. As a result, scale analysis of the β term of equation (3) results in

$$
\begin{aligned}
f_0\beta y\frac{\partial v_g}{\partial p} &\sim \frac{10^{-4}\,\mathrm{s}^{-1} \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1} \times 10^5\,\mathrm{m} \times 10\,\mathrm{m\,s}^{-1}}{1000\,\mathrm{kg\,m}^{-1}\,\mathrm{s}^{-2}} \\
&= 10^{-12}\,\mathrm{kg}^{-1}\,\mathrm{m}^2\,\mathrm{s}^{-1}
\end{aligned}
$$

The β term is therefore an order of magnitude smaller than the rest of the terms and is subsequently disregarded. This result leads to

$$
\begin{aligned}
\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) &= -Q_2 \\
\therefore Q_2 &= -\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) \\
&= -\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right), \text{ because of the the thermal wind relationship.}
\end{aligned}
$$

However, the reduced form of equation (4) is

$$
Q_2 = \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right),
$$

which contradicts the scaled result for Q_2. The implication is that in order to address this contradiction is for either the vertical shear $\left(\frac{\partial u_g}{\partial p}\right)$ or the temperature gradient $\left(\frac{\partial T}{\partial y}\right)$ to vanish. We can therefore not ignore the ageostrophic wind terms, and so the ageostrophic circulation is required to keep the flow in approximate thermal wind balance.

The role of ageostrophic circulation in vertical motion is implied through the determination of the omega (ω) motion field, since ω is determined only by the ageostrophic part of the wind field

$$
\begin{aligned}
\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} &= -\frac{\partial \omega}{\partial p} \\
\overline{\nabla}\cdot\overline{V}_a &= -\frac{\partial \omega}{\partial p}
\end{aligned}
$$

However, the total ageostrophic flow field cannot be determined by the divergence alone, because

$$
\frac{D_g\overline{V}_g}{Dt} = -f_0\overline{k}\times\overline{V}_a - \beta y\overline{k}\times\overline{V}_g
$$

By again neglecting the β effect for simplicity

$$
\begin{aligned}
\frac{D_g\overline{V}_g}{Dt} &= -f_0\overline{k}\times\overline{V}_a \\
\overline{k}\times\overline{k}\times\overline{V}_a &= -\frac{1}{f_0}\overline{k}\times\frac{D_g\overline{V}_g}{Dt}
\end{aligned}
$$

The left-hand side is

$$
\begin{aligned}
\overline{k} \times \overline{k} \times \overline{V}_a &= \overline{k} \times \overline{k} \times (V_{a1}\overline{i} + V_{a2}\overline{j}) \\
&= \overline{k} \times \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & 1 \\ V_{a1} & V_{a2} & 0 \end{vmatrix} \\
&= \overline{k} \times (\overline{i}(-V_{a2}) - \overline{j}(-V_{a1})) \\
&= -V_{a2}\overline{j} - (-\overline{i})(-V_{a1}) \\
&= -(V_{a1}\overline{i} + V_{a2}\overline{j}) \\
&= -\overline{V}_a \\
\therefore -\overline{V}_a &= -\frac{1}{f_0}\overline{k} \times \frac{D_g\overline{V}_g}{Dt} \\
\therefore \overline{V}_a &= \frac{1}{f_0}\overline{k} \times \left(\frac{\partial \overline{V}_g}{\partial t} + \overline{V}_g \cdot \overline{\nabla V}_g\right) \\
&= \frac{1}{f_0}\overline{k} \times \frac{\partial \overline{V}_g}{\partial t} + \frac{1}{f_0}\overline{k} \times \overline{V}_g \cdot \overline{\nabla V}_g
\end{aligned}
$$

The ageostrophic wind forcing therefore consists of two parts. The first term on the right represents the isallobaric[1] wind, and the second term is called the advective part of the ageostrophic wind.

Consider the isallobaric term

$$
\begin{aligned}
\frac{1}{f_0}\overline{k} \times \frac{\partial}{\partial t}\overline{V}_g &= \frac{1}{f_0}\overline{k} \times \frac{\partial}{\partial t}\left(\frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi\right) \\
&= \frac{1}{{f_0}^2}\overline{k} \times \overline{k} \times \overline{\nabla}\left(\frac{\partial \Phi}{\partial t}\right) \\
&= \frac{1}{{f_0}^2}\overline{k} \times \overline{k} \times \overline{\nabla}\chi \\
&= \frac{1}{{f_0}^2}\overline{k} \times \overline{k} \times \left(\frac{\partial \chi}{\partial x}\overline{i} + \frac{\partial \chi}{\partial y}\overline{j}\right) \\
&= \frac{1}{{f_0}^2}\overline{k} \times \left(\frac{\partial \chi}{\partial x}\overline{j} - \frac{\partial \chi}{\partial y}\overline{i}\right) \\
&= \frac{1}{{f_0}^2}\left(-\frac{\partial \chi}{\partial x}\overline{i} - \frac{\partial \chi}{\partial y}\overline{j}\right) \\
&= -\frac{1}{{f_0}^2}\overline{\nabla}\chi
\end{aligned}
$$

Therefore, the isallobaric wind is proportional to the gradient of the geostrophic tendency. Since ${f_0}^2$ is involved, there is no change of sign in crossing the equator. This isallobaric wind blows towards falling geopotential in both hemispheres.

Next consider the term that is the advective part of the ageostrophic wind. At the synoptic scale, baroclinic waves grow in the mid-latitudes due to baroclinic instability (arising from vertical shear of the mean

[1] of equal or constant pressure change.

flow and thermal wind). When such waves are part of the jet stream, the advective term is dominated by zonal advection, since jet streams are quasi-horizontal with maximum winds embedded in the mid-latitudes westerlies. Let $\overline{u}$ denote the mean zonal flow, then

$$
\begin{aligned}
\overline{V}_g \cdot \overline{\nabla V}_g &\simeq (\overline{u}\bar{i} + 0\bar{j}) \cdot \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j}\right)\overline{V}_g \\
&= \overline{u}\frac{\partial}{\partial x}\overline{V}_g \\
\therefore \frac{1}{f_0}\overline{k} \times \left(\overline{u}\frac{\partial}{\partial x}\overline{V}_g\right) &= \frac{1}{f_0}\overline{u}\frac{\partial}{\partial x}(\overline{k} \times \overline{V}_g) \\
&= \frac{1}{f_0}\overline{u}\frac{\partial}{\partial x}\left(\overline{k} \times \left(\frac{1}{f_0}\overline{k} \times \overline{\nabla}\Phi\right)\right) \\
&= \frac{1}{{f_0}^2}\overline{u}\frac{\partial}{\partial x}\left(\overline{k} \times \left(\overline{k} \times \left(\frac{\partial \Phi}{\partial x}\bar{i} + \frac{\partial \Phi}{\partial y}\bar{j}\right)\right)\right) \\
&= \frac{1}{{f_0}^2}\overline{u}\frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial x}\bar{i} - \frac{\partial \Phi}{\partial y}\bar{j}\right) \\
&= -\frac{1}{{f_0}^2}\overline{u}\frac{\partial}{\partial x}(\overline{\nabla}\Phi)
\end{aligned}
$$

$\implies$ The ageostrophic wind

$$\overline{V}_a = -\frac{1}{{f_0}^2}\left[\overline{\nabla}\chi + \overline{u}\frac{\partial}{\partial x}(\overline{\nabla}\Phi)\right]$$

Next, we will perform a scale analysis on this ageostrophic wind equation. First, do the scale analysis of the isallobaric wind

$$
\begin{aligned}
\overline{V}_{\text{isall}} &= -\frac{1}{{f_0}^2}\overline{\nabla}\left(\frac{\partial \Phi}{\partial t}\right) \\
&\sim -\frac{1}{f^2}\frac{1}{L}\left(\frac{L}{U}\right)^{-1}\delta\Phi \\
&= -\frac{1}{f^2}\frac{U}{L^2}\left(-\frac{1}{\rho}\delta p\right) \\
&= \frac{U\delta p}{f^2 L^2 \rho} \\
&\sim \frac{10\,\mathrm{m\,s^{-1}} \times \left(10 \times 10^2\,\mathrm{kg\,m\,s^{-2}\,m^{-2}}\right)}{10^{-8}\,\mathrm{s^{-2}} \times (10^6\,\mathrm{m})^2 \times 1\,\mathrm{kg\,m^{-3}}} \\
&= \frac{10^4}{10^4}\frac{\mathrm{s^{-3}}}{\mathrm{s^{-2}\,m^{-1}}} = 1\,\mathrm{m\,s^{-1}}
\end{aligned}
$$

Next, scale the term that is the advective part of the ageostrophic wind

$$
\begin{aligned}
& -\frac{1}{{f_0}^2}\overline{u}\frac{\partial}{\partial x}\overline{\nabla}\Phi \\
& \sim -\frac{1}{f^2}U\frac{1}{L}\frac{1}{L}\delta\Phi \\
& = -\frac{1}{f^2}\frac{U}{L^2}\left(-\frac{1}{\rho}\delta p\right) \\
& = \frac{U\delta p}{f^2L^2\rho} \quad \text{, which is the same as was found for the isallobaric wind.}
\end{aligned}
$$

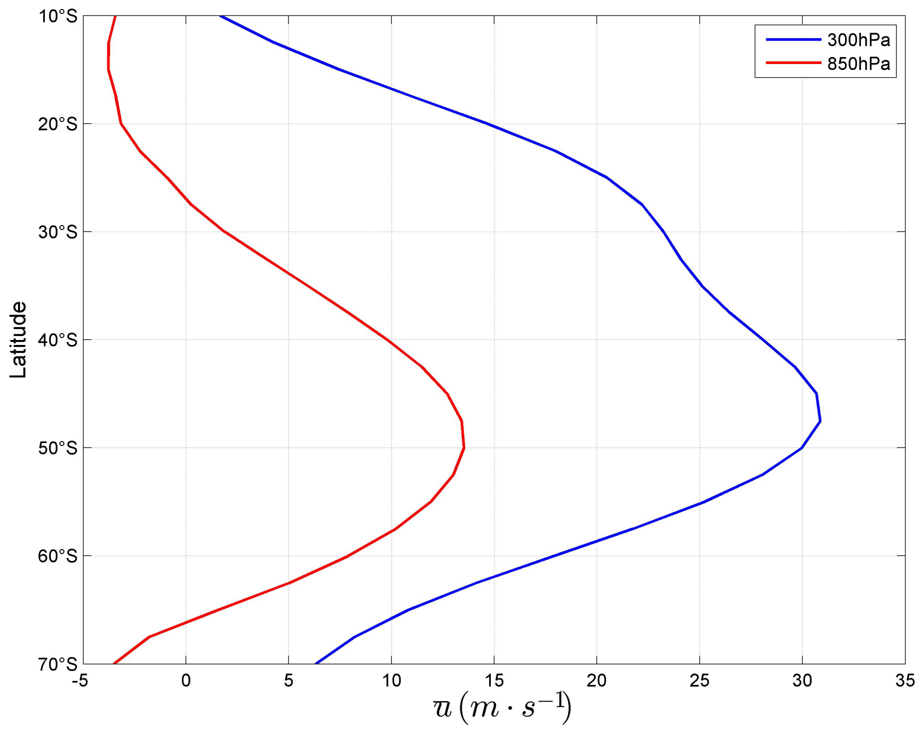


Figure 4: Mean zonal flow distribution at the two pressure levels indicated.

The scale analysis done here shows that both the isallobaric wind and the advective part are about $1\,\mathrm{m\,s^{-1}}$, given both the typical horizontal wind speed and the mean zonal flow to be $10\,\mathrm{m\,s^{-1}}$. However, profiles of the time-mean zonal geostrophic wind, averaged over longitudes, show that for two isobaric levels, one at $850\,\mathrm{hPa}$ and the other at $300\,\mathrm{hPa}$, that there is a strong jetstream at $300\,\mathrm{hPa}$ in the mid-latitudes.

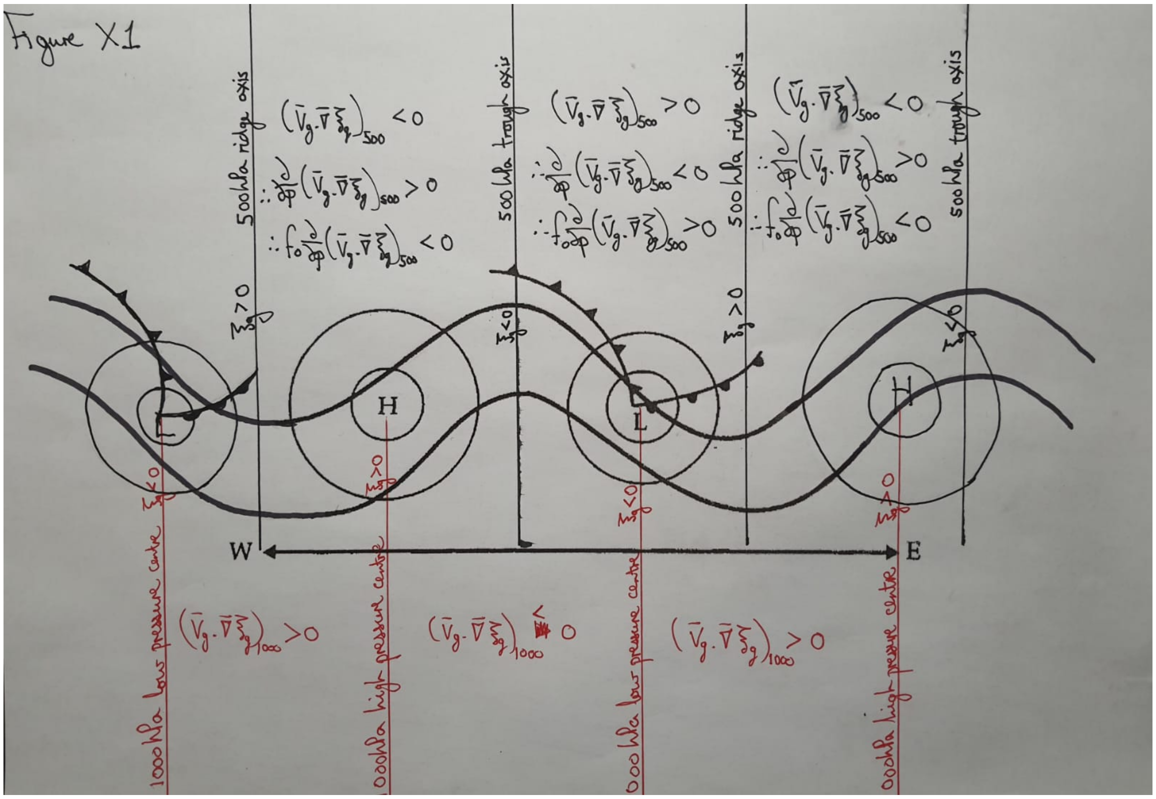
Zonal mean winds of the $300\,\mathrm{hPa}$ jetstream are typically of the order of 30 to $40\,\mathrm{m\,s^{-1}}$ in the mid-latitudes, while at $850\,\mathrm{hPa}$ the zonal maximum wind is closer to the $10\,\mathrm{m\,s^{-1}}$ value used in the scale analysis. Therefore, at the $300\,\mathrm{hPa}$ jetstream the advective contribution to the ageostrophic wind dominates over the isallobaric contribution. At both high and low latitudes, that is on the edges or flanks of mid-latitude baroclinic systems, zonal wind at the $300\,\mathrm{hPa}$ level are of similar strength and weak. Thus, at the flanks the two contributions to the ageostrophic flow are similarly small so that the resulting ageostrophic wind is small. The net effect is that at $300\,\mathrm{hPa}$ the ageostrophic motion is primarily zonal.

At the $850\,\mathrm{hPa}$ level, owing to the relatively weak zonal wind in the mid-latitudes, the advective part does not dominate as it does at the $300\,\mathrm{hPa}$ level. Moreover, owing to the weak wind at $850\,\mathrm{hPa}$ at the flanks, the advective contribution is nearly zero there. On the other hand, since the isallobaric wind is always directed down the gradient of pressure tendency, it will cause a meridional component in both the northern and southern flanks of the baroclinic wave, and a zonal component along the central axis of the wave. However, along the zonal axis the isallobaric contribution may balance that of the advective contribution, with the net result that the ageostrophic motion will only have a meridional component in the northern and southern flanks at $850\,\mathrm{hPa}$.

Vertical and horizontal motions in a developing short-wave baroclinic system in the Southern Hemisphere – a summary

This chapter on the quasi-geostrophic theory started off by presenting a classic theory on the motion of mid-latitude developing baroclinic systems in the Southern Hemisphere. In this theory the atmospheric level of non-divergence (transition from positive to negative divergence, and vice versa) was introduced and it was concluded that if this level is low enough in altitude that the system will move eastward. The eastward movement of these systems fits into the theory of short-wave developing baroclinic systems discussed in this chapter. The $500\,\mathrm{hPa}$ level is often assumed to be the level of non-divergence, and is about halfway through the vertical depth of the mass of the atmosphere. Next, through the use of highly idealistic circulation fields at the $500\,\mathrm{hPa}$ level the relative importance of the advections of relative and planetary vorticity was investigated – relative vorticity advection dominates planetary vorticity advection for short-wave systems. Moreover, for fast-moving extratropical short-wave weather systems that are not rapidly amplifying or decaying the local rate of change of geostrophic vorticity is represented only by the advection of geostrophic vorticity. However, in the presence of developing systems this rate is also a function of the divergence effect, which forms part of the ageostrophic flow.

Consider Figure X1 that shows a mean sea-level (i.e., $1000\,\mathrm{hPa}$) pressure pattern represented by fine lines and $500\,\mathrm{hPa}$ pattern by thick lines for the Southern Hemisphere. First we will consider vorticity advection for short-wave systems at both the $1000\,\mathrm{hPa}$ and $500\,\mathrm{hPa}$ levels, by neglecting the effect of planetary vorticity advection. The vertical lines on the figure respectively represent trough and ridge axes at $500\,\mathrm{hPa}$ as well as at the surface. At the surface by the centres of both the low and the high pressure systems, the geostrophic vorticity advection is close to zero. However, at the $500\,\mathrm{hPa}$ level above the surface low (high) pressure system the advection of geostrophic vorticity is higher (lower) and positive (negative). Vorticity advection is usually higher in absolute terms at $500\,\mathrm{hPa}$ than at the surface because the wind speeds tend to increase with height, therefore $500\,\mathrm{hPa}$ winds near a trough will often be stronger than low-level winds. Accordingly, there is a positive change in the vertical of the geostrophic vorticity advection, referred to as differential vorticity advection, which is a negative value above the surface low owing to the decrease in pressure with increasing height above the surface. However, in the Southern Hemisphere where the Coriolis parameter is negative, multiplying the differential vorticity advection term with f_0 leads to a positive differential vorticity advection term. Take note that rising air is implied by an increase with height of cyclonic relative vorticity advection, which is the case above the surface low.



The positive advection of relative vorticity at the $500\,\mathrm{hPa}$ level above the surface low pressure system has implications for the horizontal displacement of mid-latitude disturbances. Consider Figure X2 as well as the geopotential tendency equation. This equation consists of two terms on the right respectively representing vorticity advection (the dominant forcing term in the upper troposphere) and thickness or temperature advection (largest in magnitude in the lower troposphere). Due to the advection of relative vorticity at $500\,\mathrm{hPa}$ above the surface low, geopotential heights are decreasing, while geopotential heights are increasing at $500\,\mathrm{hPa}$ above the surface low. Vorticity advection at the $500\,\mathrm{hPa}$ level thus acts to propagate the disturbance horizontally to spread it vertically. Regarding the temperature advection term of the tendency equation, below the $500\,\mathrm{hPa}$ trough cold advection in association with the cold front occurs, while warm advection in association with the warm front occurs below the $500\,\mathrm{hPa}$ ridge. The effect of cold advection below the $500\,\mathrm{hPa}$ trough is to deepen the trough in the upper troposphere, while the effect of the warm advection below the $500\,\mathrm{hPa}$ ridge is to build the ridge in the upper troposphere. What is the source of this advection? Recalling the figure of a developing synoptic-scale system in the section that discussed the Sutcliffe form of the omega equation, the $500\,\mathrm{hPa}$ contours lead the $1000\,\mathrm{hPa}$ contours due to the westward tilt of the developing system. During this development the result is that the $500\,\mathrm{hPa}$ geopotential field leads the isotherm pattern. While the angle between the geopotential height contours and the thickness contours increases, an increase in the horizontal temperature advection is the result. However, as the system is allowed to further develop, the surface low pressure contours, the $500\,\mathrm{hPa}$ contours and the thickness contours come into alignment with each other. This later stage of development results in the weakening of the horizontal temperature advection and marks the end of the intensification phase in the lifecycle of the short-wave baroclinic system.

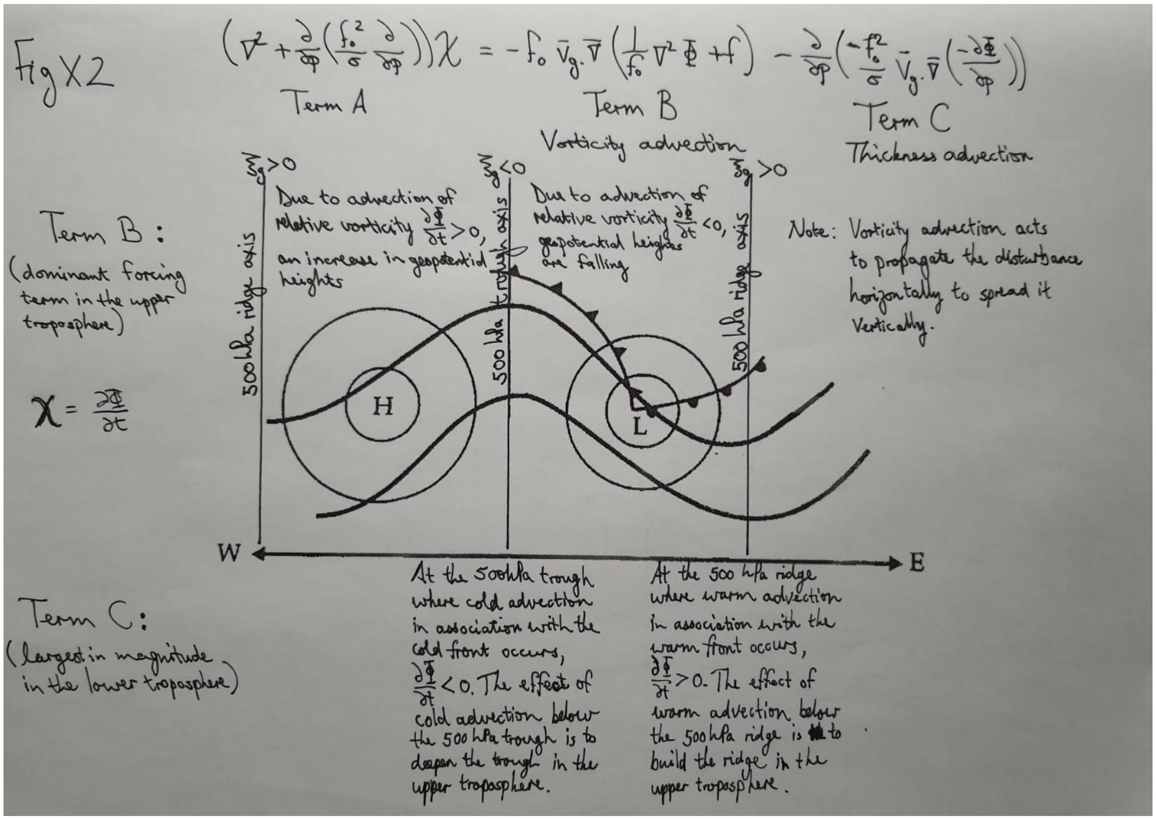


Fig X2
$\left(\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right)\chi = -f_0\,\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right) - \frac{\partial}{\partial p}\left(-\frac{f_0^2}{\sigma}\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right)$
Term A
Term B
Vorticity advection
Term C
Thickness advection
Term B :
(dominant forcing term in the upper troposphere)
$\chi = \frac{\partial\Phi}{\partial t}$
$\zeta_g>0$
$\zeta_g<0$
$\zeta_g>0$
Due to advection of relative vorticity $\frac{\partial\Phi}{\partial t}>0$, an increase in geopotential heights
Due to advection of relative vorticity $\frac{\partial\Phi}{\partial t}<0$, geopotential heights are falling
Note: Vorticity advection acts to propagate the disturbance horizontally to spread it vertically.
500 hPa ridge axis
500 hPa trough axis
500 hPa ridge axis
H
L
W
E
Term C:
(largest in magnitude in the lower troposphere)
At the 500hPa trough where cold advection in association with the cold front occurs, $\frac{\partial\Phi}{\partial t}<0$. The effect of cold advection below the 500 hPa trough is to deepen the trough in the upper troposphere.
At the 500 hPa ridge where warm advection in association with the warm front occurs, $\frac{\partial\Phi}{\partial t}>0$. The effect of warm advection below the 500 hPa ridge is to build the ridge in the upper troposphere.

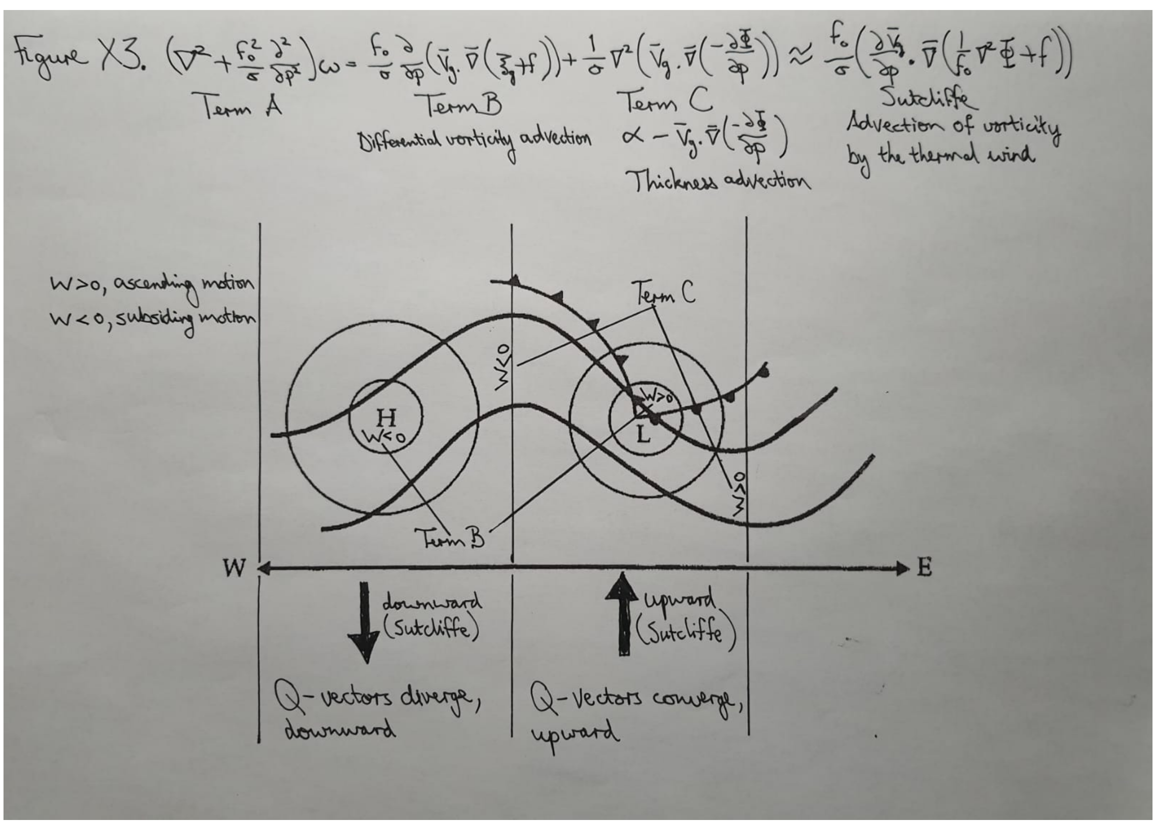


Figure X3. $\left(\nabla^2+\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\vec{\nabla}(\zeta_g+f)\right)+\frac{1}{\sigma}\nabla^2\left(\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right) \approx \frac{f_0}{\sigma}\left(\frac{\partial\vec{V}_g}{\partial p}\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\right)$
Term A
Term B
Differential vorticity advection
Term C
$\propto -\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)$
Thickness advection
Sutcliffe
Advection of vorticity by the thermal wind
W>0, ascending motion
W<0, subsiding motion
Term C
W<0
W>0
W>0
H
W<0
L
Term B
W
E
downward (Sutcliffe)
upward (Sutcliffe)
Q-vectors diverge, downward
Q-vectors converge, upward

The omega equation was used to determine where upward and downward motion in a developing system may occur. Figure X3 shows the results from analyzing three versions of this equation, namely its traditional form that consists of two terms on the right respectively representing differential vorticity advection and thickness (temperature) advection, the reduced Sutcliffe form that only has one term on the right representing the advection of vorticity by the thermal wind, and the Q-vector form. The differential vorticity advection terms have shown that upward (downward) motion in the developing system occurs over the surface low (high) pressure systems, and that the thickness advection forces upward (downward) motion at the $500\,\mathrm{hPa}$ ridge (trough) axis ahead (behind) the warm (cold) front. Although the interpretations of these two physical processes have apparent advantages as demonstrated here, in practice there is often a significant amount of cancellation between them. For this reason an alternative, albeit an approximate form of the omega equation, is often applied in synoptic analyses – hence the Sutcliffe version of the equation. This version showed that upward (downward) motion is forced east (west) of the $500\,\mathrm{hPa}$ trough above the surface low (high) pressure system. This finding is in agreement with the interpretation of the Q-vector form of the omega equation as shown above.

To summarize, temperature advection forces the strengthening of mid-tropospheric troughs and ridges, the advection of relative vorticity acts to propagate the developing system horizontally, while differential relative vorticity advection forces rising (sinking) motion over surface low (high) pressure systems, as is the advection of vorticity by the thermal wind.