A One Line Derivation of DCC: Application of a Vector Random Coefficient Moving Average Process*

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Abstract

One of the most widely-used multivariate conditional volatility models is the dynamic conditional correlation (or DCC) specification. However, the underlying stochastic process to derive DCC has not yet been established, which has made problematic the derivation of asymptotic properties of the Quasi-Maximum Likelihood Estimators. The paper shows that the DCC model can be obtained from a vector random coefficient moving average process, and derives the stationarity and invertibility conditions. The derivation of DCC from a vector random coefficient moving average process raises three important issues: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model for DCC is based on the standardized shocks rather than the returns shocks. The derivation of the regularity conditions should subsequently lead to a solid statistical foundation for the estimates of the DCC parameters.

Keywords: Dynamic conditional correlation, dynamic conditional covariance, vector random coefficient moving average, stationarity, invertibility, asymptotic properties.

JEL classifications: C22, C52, C58, G32.

1. Introduction

Among multivariate conditional volatility models, the dynamic conditional correlation (or DCC) specification of Engle (2002) is one of the most widely used in practice. The basic DCC modelling approach has been as follows: (i) estimate the univariate conditional variances using the GARCH(1,1) model of Bollerslev (1986), which are based on the returns shocks; and (ii) estimate what is purported to be the conditional correlation matrix of the standardized residuals. The first step is entirely arbitrary as the conditional variances could just as easily be based on the standardized residuals themselves, as will be shown in Section 4 below.

A similar comment applies to the varying conditional correlation model of Tse and Tsui (2002), where the first stage is based on a standard GARCH(1,1) model using returns shocks. The second stage is slightly different from the DCC formulation as the conditional correlations are defined appropriately. However, no regularity conditions are presented, and hence no statistical properties are given.

The DCC model has been analyzed critically in a number of papers as its underlying stochastic process has not yet been established, which has made problematic the derivation of the asymptotic properties of the Quasi-Maximum Likelihood Estimators (QMLE). To date, the statistical properties of the QMLE of the DCC parameters have been derived under highly restrictive and unverifiable regularity conditions, which in essence amounts to proof by assumption.

This paper shows that the DCC specification can be obtained from a vector random coefficient moving average process, and derives the conditions for stationarity and invertibility. The derivation of regularity conditions should subsequently lead to a solid statistical foundation for the estimates of the DCC parameters.

The derivation of DCC from a vector random coefficient moving average process raises three important issues: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to

obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model for DCC is based on the standardized shocks rather than the returns shocks.

The remainder of the paper organized is as follows. In Section 2, the standard ARCH model is derived from a random coefficient autoregressive process to provide a background for the remainder of the paper. In Section 3, the DCC model is discussed. Section 4 presents a vector random coefficient moving average process, from which DCC is derived in Section 5. The conditions for stationarity and invertibility are given in Section 6. Some concluding comments are given in Section 7.

2. Random Coefficient Autoregressive Process

Consider the following a random coefficient autoregressive process of order one:

$$\varepsilon_{it} = \phi_t \varepsilon_{t-1} + \eta_t \tag{1}$$

where

$\phi_t \sim iid(0, \alpha)$,

$\eta_t \sim iid\ (0, \omega)$.

The ARCH(1) model of Engle (1982) can be derived as (see Tsay (1987)):

$$h_t = E(\varepsilon_t^2 \mid I_{t-1}) = \omega + \alpha \varepsilon_{t-1}^2 . \tag{2}$$

where h_t is conditional volatility, and I_{t-1} is the information set at time *t-1*. The use of an infinite lag length for the random coefficient autoregressive process leads to the GARCH model of Bollerslev (1986).

The scalar BEKK and diagonal BEKK models of Baba et al. (1985) and Engle and Kroner (1995) can be derived from a vector random coefficient autoregressive process (see McAleer et al. (2008)). As the statistical properties of vector random coefficient autoregressive processes are well known, the statistical properties of the parameter estimates of the ARCH, GARCH, scalar BEKK and diagonal BEKK models are straightforward to establish.

3. DCC Specification

Let the conditional mean of financial returns be given as:

$$y_t = E(y_t \mid I_{t-1}) + \varepsilon_t \tag{3}$$

where $y_t = (y_{1t,\ldots,} y_{mt})'$, y_{it} = $\Delta \log P_{it}$ represents the log-difference in stock prices (P_{it}), i = 1,…,m, I_{t-1} is the information set at time *t-1*, and ε_t is conditionally heteroskedastic. Without distinguishing between dynamic conditional covariances and dynamic conditional correlations, Engle (2002) presented the DCC specification as:

$$Q_t = (1-\alpha-\beta)\overline{Q} + \alpha \eta_{t-1}\eta_{t-1}^{'} + \beta Q_{t-1} \tag{4}$$

where $\overline{Q}$ is assumed to be positive definite with unit elements along the main diagonal, the scalar parameters are assumed to satisfy the stability condition, $\alpha + \beta <$ 1, the standardized shocks, $\eta_t = (\eta_{1t,\ldots,}\eta_{mt})'$ are given as $\eta_{it} = \varepsilon_{it} / \sqrt{h_{it}}$, with $\varepsilon_t = D_t \eta_t$, and D_t is a diagonal matrix with typical element $\sqrt{h_{it}}$, i = 1,…,m.

As the matrix in equation (4) does not satisfy the definition of a correlation matrix, Engle (2002) uses the following standardization:

$$R_t = (diag(Q_t))^{1/2} Q_t (diag(Q_t))^{1/2} \tag{5}$$

There is no clear explanation given in Engle (2002) for the standardization in equation (5) or, more recently, in Aielli (2013). The standardization in equation (5) might make sense if the matrix Q_t were the conditional covariance matrix of ε_t or η_t, though this is not made clear. Despite the title of the paper, Aielli (2013) also does not provide any stationarity conditions for the DCC model, and does not mention invertibility. Indeed, in the literature on DCC, it is not clear whether equation (4) refers to a conditional covariance or a conditional correlation matrix. Some caveats regarding DCC are given in Caporin and McAleer (2013).

4. Vector Random Coefficient Moving Average Process

Marek (2005) proposed a linear moving average model with random coefficients (RCMA), and established the conditions for stationarity and invertibility. In this section, we derive the stationarity and invertibility conditions of a vector random coefficient moving average process.

Consider a univariate random coefficient moving average process given by:

$$\varepsilon_t = \theta_t \eta_{t-1} + \eta_t \tag{6}$$

where

$$\theta_t \sim iid\ (0, \alpha).$$

The conditional and unconditional expectations of ε_t are zero. The conditional variance of ε_t is given by:

$$h_t = E(\varepsilon_t^2 \mid I_{t-1}) = \omega + \alpha \eta_{t-1}^2 \tag{7}$$

which differs from the ARCH(1) model in equation (2) in that the returns shock is replaced by the standardized shock. The use of an infinite lag length for the random coefficient moving average process in equation (6) would leads to generalized ARCH model that differs from the GARCH model of Bollerslev (1986).

The univariate ARCH(1) model in equation (7) is contained in the family of GARCH models proposed by Hentschel (1995), and the augmented GARCH model class of Duan (1997).

It can be shown seen from the results in Marek (2005) that a sufficient condition for stationarity is that the vector sequence $\upsilon_t = (\eta_t, \theta_t \eta_{t-1})'$ is stationary. Moreover, by Lemma 2.1 of Marek (2005), a sufficient condition for invertibility is that:

$$E\left[\log|\theta_t|\right] < 0 . \tag{8}$$

The stationarity of $\upsilon_t = (\eta_t, \theta_t \eta_{t-1})'$ and the invertibility condition in equation (8) are new results for the univariate ARCH(1) model given in equation (7), as well as its direct extension to GARCH models.

Extending the analysis given above to the multivariate case and to a vector random coefficient moving average (RCMA) model of order *p*, we can derive a special case of DCC(*p*,*q*), namely DCC(*p*,*0*), as follows:

$$\varepsilon_t = \sum_{j=1}^{p} \theta_{jt} \eta_{t-j} + \eta_t \tag{9}$$

where ε_t and η_t are both $m \times 1$ vectors and θ_{jt}, $j = 1,\ldots,p$ are random *iid* $m \times m$ matrices.

As $\eta_t \sim iid\ (0, \omega)$, the unconditional variance of ε_{it} is given as:

$$E(h_t) = (1+\alpha)\omega.$$

For the multivariate case in equation (9), it is assumed that the vector $\eta_t \sim iid\ (0,\Omega)$. As the diagonal elements of Ω are equal to unity, this is also the correlation matrix of η_t. It follows that:

$$E(H_t) = \left(1+\sum_{j=1}^{p}\alpha_j\right)\Omega.$$

This approach can easily be extended to include autoregressive terms. For example, in a model analogous to GARCH(*p*,*q*), namely:

$$H_t = \Omega + \sum_{i=1}^{p}\alpha_i \eta_{t-i}\eta_{t-i}^{'} + \sum_{j=1}^{q}\beta_j H_{t-j}$$

where $\beta_j \in [0,1)$ and $\sum_{j=1}^{q}\beta_j < 1$, it follows that:

$$E(H_t) = \left(\frac{1+\sum_{i=1}^{p}\alpha_i}{1-\sum_{j=1}^{q}\beta_j}\right)\Omega.$$

The derivation given above shows that, as compared with the standard DCC formulation, our formulation permits straightforward computation of the unconditional variances and covariances. It should also be noted that in Aielli's (2013) variation of the standard DCC model, it is possible to calculate the unconditional expectation of the Q_t matrix, as in equation (4), but this is not equal to the unconditional covariance matrix of ε_t, which is analytically intractable. This is an additional advantage of using the vector random coefficient moving average process given in equation (9).

5. One Line Derivation of DCC

If θ_{jt} in equation (9) is given as:

$\theta_{jt} = \lambda_{jt} I_m$, with $\lambda_{jt} \sim iid(0, \alpha_j)$, $j = 1, \ldots, p,$

where λ_{jt} is a scalar random variable, then the conditional covariance matrix can be shown to be:

$$H_t = E(\varepsilon_t \varepsilon_t^{'} \mid I_{t-1}) = \Omega + \sum_{j=1}^{p} \alpha_j \eta_{t-j} \eta_{t-j}^{'} \ . \tag{10}$$

The DCC model in equation (4) is obtained by letting $p \to \infty$ and standardizing H_t to obtain a conditional correlation matrix. For the case p=1 in equation (10), the appropriate univariate conditional volatility model is given in equation (7), which uses the standardized shocks, rather than in equation (2), which uses the returns shocks.

The derivation of DCC in equation (10) from a vector random coefficient moving average process is important as it: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model for DCC is be based on the standardized shocks rather than the returns shocks.

6. Derivation of Stationarity and Invertibility

This section derives the stationarity and invertibility conditions for the DCC model.

Assumption 1. $E\left[\log\left\|\Theta_{t-k}\right\|\right] < \sqrt{pm}$ (11)

where $\|\Theta_t\|$ is the Frobenius norm, and Θ_t is given by:

$$\Theta_t = \begin{pmatrix} -\theta_{1t} & -\theta_{2t} & \dots & -\theta_{pt} \\ 1 & 0 & \dots & 0 \\ . & . & . & . \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

Theorem 1. A sufficient condition for stationarity is that the vector sequence:

$$\upsilon_t = (\eta_t, \theta_{1t}\eta_{t-1}, \dots, \theta_{pt}\eta_{t-p})'$$

is stationary. Furthermore, under Assumption 1, the vector random coefficient moving average process, ε_t, is invertible.

Proof: The proof of stationarity is similar to that given above for the univariate random coefficient moving average process. For invertibility, note that:

$$\eta_t = \varepsilon_t - \sum_{j=1}^{p} \theta_{jt}\eta_{t-j}$$

which can be written as:

$$\tilde{\eta}_t = \Theta_t \tilde{\eta}_{t-1} + \tilde{\varepsilon}_t$$

where

$$\tilde{\eta}_t = (\eta_t, \eta_{t-1}, \dots, \eta_{t-p+1})' \text{ and } \tilde{\varepsilon}_t = (\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-p+1})'.$$

Hence,

$$\tilde{\eta}_t = \sum_{j=0}^{n} \prod_{k=1}^{j} \Theta_{t-k+1} \varepsilon_{t-j} + \prod_{k=1}^{n-1} \Theta_{t-k} \eta_{t-n} .$$

Now let:

$$\tilde{\eta}_t^{(n)} = \sum_{j=0}^{n} \prod_{k=1}^{j} \Theta_{t-k+1} \varepsilon_{t-j}$$

Consider

$$\frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \eta_t - \eta_t^n \right\| = \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \prod_{k=1}^{n-1} \Theta_{t-k} \eta_{t-n} \right\|$$

$$\leq \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \prod_{k=1}^{n-1} \Theta_{t-k} \right\| + \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \eta_{t-n} \right\|$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{\sqrt{pm}} \left\| \Theta_{t-k} \right\| + \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \eta_{t-n} \right\|$$

$$\xrightarrow[a.s.]{} E \log \frac{1}{\sqrt{pm}} \left\| \Theta_{t-k} \right\| < 0$$

as $E \log \left\| \Theta_{t-k} \right\| < \sqrt{pm}$, by assumption. This implies that $\eta_t - \eta_t^n \xrightarrow[a.s.]{} 0$ and, hence, η_t is asymptotically measurable with respect to { $\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ }, and ε_t is invertible. •

Note that a sufficient condition for equation (11) is that:

$$\sum_{j=1}^{p} E \left\| \theta_{jt} \right\|^2 < m \tag{12}$$

as $\quad E\log\frac{1}{\sqrt{pm}}\|\Theta_{t-k}\| \le \log E\frac{1}{\sqrt{pm}}\|\Theta_{t-k}\|$

$$= \log E\frac{1}{\sqrt{pm}}\sqrt{\sum_{j=1}^{p}\|\theta_{jt}\|^2 + (p-1)m}$$

$$= \log E\sqrt{\frac{1}{pm}\sum_{j=1}^{p}\|\theta_{jt}\|^2 + (p-1)/p}$$

$$\le \log\sqrt{\frac{1}{pm}\sum_{j=1}^{p}E\|\theta_{jt}\|^2 + (p-1)/p}$$

$< \ 0$.

The condition given in equation (12) may be easier to check that that in equation (11).

For the special case $\theta_{jt} = \lambda_{jt} I_m$, with $\lambda_{jt} \sim iid(0,\alpha_j)$, $\quad j = 1, \ldots, p,$ discussed in Section 5 above, the condition in equation (12) simplifies to the wel-known condition on the long-run persistence to returns shocks, namely:

$$\sum_{j=1}^{p} E\lambda_{jt}^2 = \sum_{j=1}^{p}\alpha_j < 1.$$

7. Conclusion

The paper is concerned with one of the most widely-used multivariate conditional volatility models, namely the dynamic conditional correlation (or DCC) specification. As the underlying stochastic process to derive DCC has not yet been established, the paper showed that the DCC specification could be obtained from a vector random coefficient moving average process, and

derived the stationarity and invertibility conditions. The derivation of the regularity conditions should eventually lead to a solid foundation for the statistical analysis of the estimates of the DCC parameters.

The derivation of DCC from the vector random coefficient moving average process demonstrated that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model. Moreover, the derivation provided the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model. Finally, the derivation also showed that the appropriate ARCH or GARCH model for DCC is based on the standardized shocks rather than the returns shocks. The derivation of regularity conditions should subsequently lead to a solid statistical foundation for the QMLE of the DCC parameters.

References

Aielli, G.P. (2013), Dynamic conditional correlations: On properties and estimation, Journal of Business and Economic Statistics, 31, 282-299.

Baba, Y., R.F. Engle, D. Kraft and K.F. Kroner (1985), Multivariate simultaneous generalized ARCH. Unpublished manuscript, Department of Economics, University of California, San Diego, CA, USA.

Bollerslev, T. (1986), Generalised autoregressive conditional heteroscedasticity, Journal of Econometrics, 31, 307-327.

Caporin, M. and M. McAleer (2013), Ten things you should know about the dynamic conditional correlation representation, Econometrics, 1(1), 115-126.

Duan, J.-C. (1997), Augmented GARCH(p,q) process and its diffusion limit, Journal of Econometrics, 79, 97-127.

Engle, R.F. (1982), Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econometrica, 50, 987-1007.

Engle, R. (2002), Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional hereoskedasticity models, Journal of Business and Economic Statistics, 20, 339-350.

Engle, R.F. and K.F. Kroner (1995), Multivariate simultaneous generalized ARCH, Econometric Theory, 11, 122-150.

Hentschel, L. (1995), All in the family: Nesting symmetric and asymmetric GARCH models, Journal of Financial Economics, 39, 71-104.

Marek, T. (2005), On invertibility of a random coefficient moving average model, Kybernetika, 41(6), 743-756.

McAleer, M., F. Chan, S. Hoti and O. Lieberman (2008), Generalized autoregressive conditional correlation, Econometric Theory, 24(6), 1554-1583.

Tsay, R.S. (1987), Conditional heteroscedastic time series models. Journal of the American Statistical Association, 82, 590-604.

Tse, Y.K. and A.K.C. Tsui (2002), A multivariate GARCH model with time-varying correlations, Journal of Business and Economic Statistics, 20, 351-362.